Dynamic Meteorology (EES 851)
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Chapter 3 Introduction to Atmospheric Thermodynamics

3.1 Derivation of the Thermodynamic Equation

(Equation editor: $D/Dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y + w\partial/\partial z$)

The thermodynamics equation is based on:

(a) Equation of state for ideal gas

$$p\alpha = RT \tag{1}$$

where α is $1/\rho$ is called specific volume (i.e. volume per unit mass).

(b) First law of thermodynamics

$$c_v dT + p d\alpha = dq \quad \text{or} \quad c_p dT - \alpha dp = dq$$

$$c_p \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} = \frac{Dq}{Dt} = J \tag{2}$$

Taking total differentiation of (1) and then substitute it into (2) leading to the thermodynamic equation:

$$c_p \frac{DT}{Dt} - \frac{RT}{p}\frac{Dp}{Dt} = \frac{Dq}{Dt} \tag{2.42}$$

or

$$c_p \frac{D\ln T}{Dt} - R\frac{D\ln p}{Dt} = \frac{1}{T}\frac{Dq}{Dt} = \frac{J}{T} = \frac{Ds}{Dt} \tag{2.43}$$

where *s* is the entropy, defined as $ds=dq/T$.

Eq. (2.42) may also be written in an alternative form,

$$\frac{DT}{Dt} - \frac{RT}{c_p p}\frac{Dp}{Dt} = \frac{1}{c_p}\frac{Dq}{Dt} \quad \text{or}$$

$$\frac{D\ln T}{Dt} - \frac{R}{c_p}\frac{D\ln p}{Dt} = \frac{1}{c_p T}\frac{Dq}{Dt} = \frac{J}{c_p T} = \frac{1}{c_p}\frac{Ds}{Dt}$$

In isobaric coordinates, the thermodynamic equation can be written as

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + S_p\omega = \frac{1}{c_p}\frac{Dq}{Dt}$$

where $\omega \equiv Dp/Dt$ is the vertical velocity in isobaric coordinates or ω vertical motion, and $S_p = \partial T/\partial p - RT/(c_p p)$ is a stability parameter. The above equation is one form of the thermodynamic equation.

➢ Potential Temperature

For an adiabatic process ($dq = 0$ or $Dq/Dt = 0$), it can be derived

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{R/c_p}.$$

If an air parcel is moved from an initial state $(T_1, p_1) = (T, p)$ to a final state $(T_2, p_2) = (\theta,\ p_o)$ adiabatically (dq=0), then the above equation can be written as

$$\theta = T\left(\frac{p_o}{p}\right)^{R/c_p} . \tag{2.44}$$

The above equation is also called Poisson equation and θ is called potential temperature. Note that $\theta =$ constant during an adiabatic process. Thus, θ is the temperature an air parcel would have if it is displaced to 1000 mb.

Using (2.44), the thermodynamic equation, (2.43), can be rewritten as

$$\frac{D\ln\theta}{Dt} = c_p\frac{D\ln T}{Dt} - R\frac{D\ln p}{Dt} = \frac{1}{T}\frac{Dq}{Dt} = \frac{Ds}{Dt} \tag{2.46}$$

For an adiabatic process, Dq/Dt = 0, (2.46) implies $D\theta/Dt$ = 0 and Ds/Dt = 0. Thus, an adiabatic process is also an isentropic process (s = constant).

➢ Linearization of the basic equations

Consider the x-momentum equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial w} - fv + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

Now let

$$u(t,x,y,z) = U(z) + u'(t,x,y,z)$$

$$v(t,x,y,z) = V(z) + v'(t,x,y,z)$$

$$w(t,x,y,z) = w'(t,x,y,z)$$

$$\rho(t,x,y,z) = \overline{\rho}(x,y,z) + \rho'(t,x,y,z)$$

$$p(t,x,y,z) = \overline{p}(x,y,z) + p'(t,x,y,z)$$

Then, the above equation can be written as

$$\frac{\partial(U+u')}{\partial t} + (U+u')\frac{\partial(U+u')}{\partial x} + (V+v')\frac{\partial(U+u')}{\partial y} + w'\frac{\partial(U+u')}{\partial z} - f(V+v') + \frac{1}{(\overline{\rho}+\rho')}\frac{\partial(\overline{p}+p')}{\partial x} = 0$$

Assuming geostrophic flow for mean wind V and neglecting the "nonlinear" terms leads to the linear equation

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + V\frac{\partial u'}{\partial y} + U_z w' - fv' + \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x} = 0$$

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[Supplementary Reading] The equations of motion, continuity equation, and thermodynamic equation, Eqs. (2.2.1) – (2.2.5), may be linearized by partitioning the field variables (Lin 2007):

$$u(t,x,y,z) = U(z) + u'(t,x,y,z),$$

$$v(t,x,y,z) = V(z) + v'(t,x,y,z),$$

$$w(t,x,y,z) = w'(t,x,y,z),$$

$$\rho(t,x,y,z) = \overline{\rho}(x,y,z) + \rho'(t,x,y,z),$$

$$p(t,x,y,z) = \overline{p}(x,y,z) + p'(t,x,y,z),$$

$$\theta(t,x,y,z) = \overline{\theta}(x,y,z) + \theta'(t,x,y,z),$$

$$T(t,x,y,z) = \overline{T}(x,y,z) + T'(t,x,y,z),$$

$$\dot{q}(t,x,y,z) = q'(t,x,y,z), \qquad (2.2.8)$$

where capital letters and overbars represent the basic state, such as synoptic scale flow in which the mesoscale disturbances evolve, and the primes indicate perturbations, such as the mesoscale flow fields, from the basic state.

The basic state is assumed to follow Newton's second law of motion, conservation of mass, and the first law of thermodynamics. The horizontal momentum equations, (2.2.1) and (2.2.2), of the basic state lead to geostrophic balance,

$$U=-\frac{1}{f\bar{\rho}}\frac{\partial\bar{p}}{\partial y}\text{; and }V=\frac{1}{f\bar{\rho}}\frac{\partial\bar{p}}{\partial x}, \tag{2.2.9}$$

while the vertical momentum equation (2.2.3) of the basic state leads to hydrostatic balance,

$$\frac{\partial\bar{p}}{\partial z}=-\bar{\rho}g, \tag{2.2.10}$$

where $\bar{p}=\bar{\rho}R_d\bar{T}$. Equations (2.2.9) and (2.2.10) automatically imply approximately thermal wind balance for the basic state

$$U_z=-\frac{g}{f\bar{\theta}}\frac{\partial\bar{\theta}}{\partial y};\ V_z=\frac{g}{f\bar{\theta}}\frac{\partial\bar{\theta}}{\partial x}, \tag{2.2.11}$$

where $\bar{\theta}=\bar{T}\left(p_o/\bar{p}\right)^{R_d/c_p}$ and subscriptions indicate partial differentiations. Conservation of mass, (2.2.4), of the basic state leads to

$$\frac{D\bar{\rho}}{Dt}+\bar{\rho}\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}\right)=\frac{1}{c_s^2}\frac{D\bar{p}}{Dt}=\frac{1}{c_s^2}\left(U\frac{\partial\bar{p}}{\partial x}+V\frac{\partial\bar{p}}{\partial y}\right)=0, \tag{2.2.12}$$

which is consistent with the geostrophic wind relation. Conservation of the basic state thermal energy gives

$$U\frac{\partial\bar{\theta}}{\partial x}+V\frac{\partial\bar{\theta}}{\partial y}=0, \tag{2.2.13}$$

which implies no basic state thermal advection by the basic wind and will be assumed for deriving the perturbation thermodynamic equation. The left-hand side of (2.2.13) is related to the uniform heating (Q_s) in the quasi-geostrophic model, which is required to satisfy the constraint that the vertical motion field vanishes at the surface and possibly at the upper boundary for some theoretical studies (Bannon 1986). In the Eady (1949) model of baroclinic instability, this term is assumed to be 0. In fact, if one assumes $V=0$, then the above equation is automatically satisfied because $\partial\bar{\theta}/\partial x=(f\bar{\theta}/g)V_z=0$, based on the basic-state thermal wind relations. Substituting (2.2.8) with (2.2.9)-(2.2.13) into (2.2.1)-(2.2.5) and neglecting the nonlinear and viscous terms, the perturbation equations for mesoscale motions in the free atmosphere (i.e. above the planetary boundary layer) can be obtained,

$$\frac{\partial u'}{\partial t}+U\frac{\partial u'}{\partial x}+V\frac{\partial u'}{\partial y}+U_zw'-fv'+\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}=0, \tag{2.2.14}$$

$$\frac{\partial v'}{\partial t}+U\frac{\partial v'}{\partial x}+V\frac{\partial v'}{\partial y}+V_zw'+fu'+\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial y}=0, \tag{2.2.15}$$

$$\frac{\partial w'}{\partial t}+U\frac{\partial w'}{\partial x}+V\frac{\partial w'}{\partial y}-g\frac{\theta'}{\bar{\theta}}+\frac{p'}{\bar{\rho}H}+\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z}=0, \tag{2.2.16}$$

$$\frac{1}{c_s^2}\left(\frac{\partial p'}{\partial t}+U\frac{\partial p'}{\partial x}+V\frac{\partial p'}{\partial y}+\bar{\rho}f(Vu'-Uv')\right)-\frac{\bar{\rho}}{H}w'+\bar{\rho}\nabla\cdot\boldsymbol{V}'=\frac{\bar{\rho}}{c_p\bar{T}}q', \tag{2.2.17}$$

$$\left(\frac{\partial\theta'}{\partial t}+U\frac{\partial\theta'}{\partial x}+V\frac{\partial\theta'}{\partial y}\right)+\frac{f\bar{\theta}}{g}(V_z u'-U_z v')+\frac{N^2\bar{\theta}}{g}w'=\frac{\bar{\theta}}{c_p\bar{T}}q'. \qquad (2.2.18)$$

Remember that N is the Brunt-Vaisala (buoyancy) frequency and H is the scale height. The Brunt-Vaisala frequency and scale height are defined, respectively, as

$$N^2 \equiv \frac{g}{\bar{\theta}}\frac{\partial\bar{\theta}}{\partial z},\quad H \equiv \frac{c_s^2}{g}, \qquad (2.2.19)$$

where

$$c_s^2 = \gamma R_d \bar{T},\ \gamma = \frac{c_p}{c_v}.$$

Note that the scale height has also been defined in the literature as the height at which the basic density of the air at surface (ρ_s) is reduced to its e-folding value, i.e. $\rho(z=H)=\rho_s e^{-1}$, assuming the air density decreases with height exponentially.

In deriving (2.2.14) and (2.2.15), we have assumed $|\rho'/\bar{\rho}| << |u'/U|$ and $|\rho'/\bar{\rho}| << |v'/V|$, which is a good first approximation in the real atmosphere. The sum of the fourth and fifth terms of 2.2.16 represents the buoyancy force ($g\rho'/\bar{\rho}$) associated with the atmospheric motion, which may also be written as. $g\rho'/\bar{\rho} = -g\theta'/\bar{\theta} + p'/(\bar{\rho}H)$ This relation reduces to $\rho'/\bar{\rho} \approx -\theta'/\bar{\theta}$ for an incompressible or Boussinesq fluid. The incompressible and Boussinesq approximations will be discussed later. The Brunt-Vaisala (buoyancy) frequency represents the natural oscillation frequency of an air parcel displaced vertically from its equilibrium position by the buoyancy force in a stably stratified atmosphere (i.e. $N^2 > 0$). The vertical oscillation period for parcels in this type of atmosphere is $2\pi/N$. In deriving Eq. (2.2.17), we have substituted the equation of state and the first law of thermodynamics, $D(\ln\theta)/Dt = \dot{q}/(c_p T)$, into the total derivative of the air density ($D\rho/Dt$). This yields the diabatic term on the right side of Eq. (2.2.17), which may be neglected for an incompressible or Boussinesq fluid since it is of the same order as other $\bar{c}_s^2$ terms for most mesoscale flows.

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3.2 Basic Concepts

➢ Dry Adiabatic Lapse Rate

$$\theta = T\left(\frac{p_o}{p}\right)^{R/c_p}. \qquad (2.44)$$

Taking $\partial(\ln(2.44))/\partial z$ (Poisson equation) leads to

$$\frac{T}{\theta}\frac{\partial \theta}{\partial z} = \frac{\partial T}{\partial z} - \frac{RT}{c_p p}\frac{\partial p}{\partial z} = \frac{\partial T}{\partial z} + \frac{g}{c_p} \qquad (2.47)$$

In deriving (2.47), we have assumed a hydrostatic atmosphere.

For an adiabatic process (θ = constant), (2.47) gives the dry adiabatic lapse rate

$$\Gamma_d = -\frac{\partial T}{\partial z} = \frac{g}{c_p}\,. \qquad (2.48)$$

Thus, the temperature decreases with height at a rate of 9.8 K/km. Note that the actual lapse rate is smaller than g/c_p, such as 5~6 K/km.

➢ Static Instability

The concept of static instability can be also understood by applying parcel theory to the vertical momentum equation,

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g\,, \qquad (7.3.1)$$

where ρ and p are the density and pressure of the air parcel, respectively.

In the *parcel theory*, we assume that:
(a) The pressure of the air parcel adjusts immediately to the pressure of its environment ($\overline{p}$), i.e. $p = \overline{p}$, when it moves away from its initial level;
(b) The environment of the air parcel is in hydrostatic balance;
(c) No compensating motions exist in the parcel's environment;

(d) The air parcel does not mix with its environment and so retains its original identity.

Applying condition (a) to the air parcel leads to

$$\frac{Dw}{Dt} = g\left(\frac{\bar{\rho}-\rho}{\rho}\right) \equiv b, \tag{7.3.2}$$

where b is the *buoyancy*, or more precisely the buoyancy force per unit mass.

The above equation indicates that the vertical acceleration of the air parcel is controlled by the *buoyancy force*, $g(\bar{\rho}-\rho)/\rho$.

It can be derived that

$$\frac{\bar{\rho}-\rho}{\rho} \approx \frac{\theta-\bar{\theta}}{\bar{\theta}} \approx \frac{T-\bar{T}}{\bar{T}}. \tag{7.3.3}$$

Substituting (7.3.3) into (7.3.2) leads to

$$\frac{Dw}{Dt} = g\frac{\bar{\rho}-\rho}{\rho} \approx g\frac{\theta-\bar{\theta}}{\bar{\theta}} \approx g\frac{T-\bar{T}}{\bar{T}} = b = \text{buoyancy} \tag{7.3.2'}$$

Note that if

$b > 0$, then the air parcel will accelerate upward

(e.g., when air parcel is warmer than its environment)

$b = 0$, then the air parcel experience no acceleration

(e.g., when air parcel and its environment has the same temperature)

$b < 0$, then the air parcel will accelerate downward

(e.g., air parcel is colder than its environment)

The Brunt-Vaisala frequency, defined as

$$N^2 \equiv \frac{g}{\overline{\theta}} \frac{\partial \overline{\theta}}{\partial z} \qquad (2.52)$$

is a measure of static stability. The overbar means the value of a variable (θ in this case) in the environment of an air parcel. It can also be shown

$$N^2 \approx -\frac{g}{\overline{\rho}} \frac{\partial \overline{\rho}}{\partial z}.$$

Equation (2.47)

$$\frac{T}{\overline{\theta}} \frac{\partial \overline{\theta}}{\partial z} = \frac{\partial T}{\partial z} + \frac{g}{c_p} \qquad (2.47)$$

can be rewritten as

$$\frac{T}{\overline{\theta}} \frac{\partial \overline{\theta}}{\partial z} = \Gamma_d - \Gamma. \qquad (2.49)$$

Based on (2.49) and (2.52) (i.e., $N^2 \equiv (g/\overline{\theta})(\partial \overline{\theta}/\partial z)$), the atmosphere is

Statically stable (static stability) if $\Gamma < \Gamma_d$ or $N^2 > 0$

Statically neutral (static neutrality) if $\Gamma = \Gamma_d$ or $N^2 = 0$

Statically unstable (static instability) if $\Gamma > \Gamma_d$ or $N^2 < 0$.

where

$$N^2 \equiv \frac{g}{\overline{\theta}} \frac{\partial \overline{\theta}}{\partial z}, \qquad (2.52)$$

is the Brunt-Vaisala frequency.

➢ Conditional Instability and Thermodynamic Diagram

The necessary conditions for conditional instability to occur are: (a) $\Gamma_s < \gamma < \Gamma_d$ and (b) a lifting of the air parcel past its LFC. A thermodynamic diagram can help understand the concept and determine conditionally stability or instability.

- Skew-T log-p Diagram [reading assignment]
  (Click here for: Skew-T, log-p diagram analysis procedure)
  (Click here for: interactive Skew-T diagram)

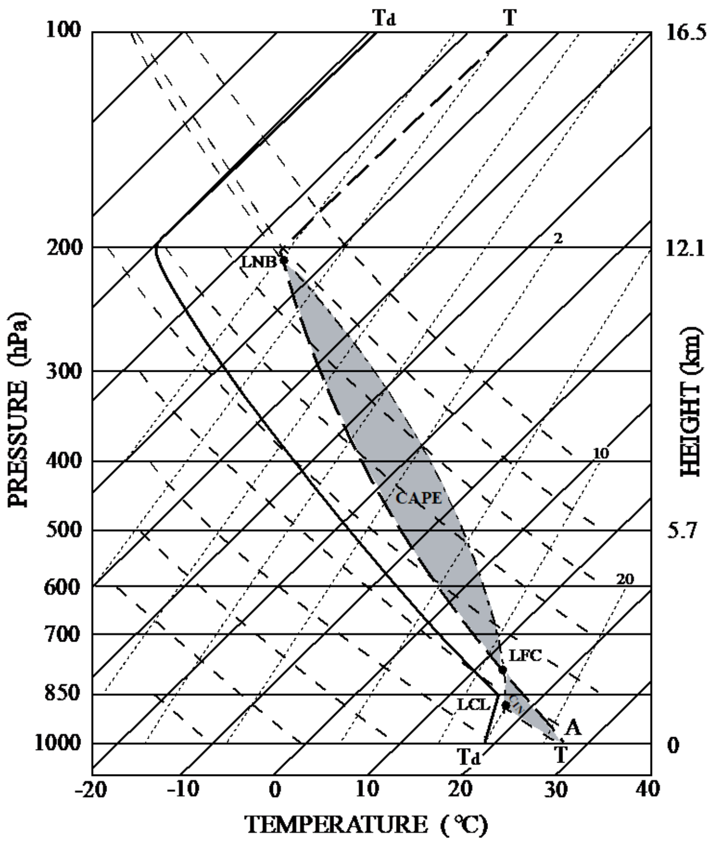

Fig. 7.5: Example of a sounding with conditional instability displayed on a *skew-T log-p* thermodynamic diagram. The lifting condensation level (LCL), level of free convection (LFC), and level of neutral buoyancy (LNB) for the air parcel originated at A are denoted in the figure. The convective available potential energy (CAPE) is the area enclosed by the temperature curve (thick dashed line) and moist adiabat (dot-dashed curve) in between LFC and LNB, while the convective inhibition is the area enclosed the temperature curve and dry adiabat (below LCL) and moist adiabat (above LCL) in between the surface and the LFC. (From Lin 2007, Mesoscale Dynamics)

- Need to learn how to:

  (a) Use skew-T log-p diagram to find temperature for an air parcel located at 500 mb on a sounding.
  (b) Given a sounding, (T, T_d), on the thermodynamic diagram, estimate the LCL (lifting condensation level) for an air parcel lifted upward from the ground. LCL is where the cloud base is located.
  (c) Determine LFC (Level of Free Convection), LNB (Level of Neutral Buoyancy), CAPE (Convective Available Potential Energy), and positive area.

3.3 Scale Analysis of Thermodynamic Energy Equation

One form of the thermodynamic equation derived in 3.2 can be written as

$$c_p \frac{D\ln\theta}{Dt} = \frac{1}{T}\frac{Dq}{Dt} = \frac{Ds}{Dt} = \frac{J}{T}, \qquad (2.46)$$

Equation (2.46) can be linearized by letting

$$\theta(x,y,z,t) = \theta_o(z) + \theta'(x,y,z,t)$$

and assuming

$$|\theta'/\theta_o| << 1, \quad |d\theta'/dz| << |d\theta_o/dz|,$$
$$\ln\theta = \ln[\theta_o(1+\theta'/\theta_o)] \approx \ln\theta_o + \theta'/\theta_o,$$

to be

$$\frac{1}{\theta_o}\left(\frac{\partial\theta'}{\partial t} + u\frac{\partial\theta'}{\partial x} + v\frac{\partial\theta'}{\partial y}\right) + w\left(\frac{1}{\theta_o}\frac{\partial\theta_o}{\partial z}\right) = \frac{J}{c_p T} \qquad (2.53)$$

where $J = Dq/Dt$ is the heating rate. The above equation may also be written as

$$\frac{\partial\theta'}{\partial t} + u\frac{\partial\theta'}{\partial x} + v\frac{\partial\theta'}{\partial y} + \left(\frac{N^2\theta_o}{g}\right)w = \left(\frac{\theta_o}{c_p T}\right)J\,, \qquad (2.53)'$$

where N is the Brunt-Vaisala (buoyancy) frequency defined as $N^2 = (g/\theta)(\partial\theta/\partial z) \approx (g/\theta_o)(\partial\theta_o/\partial z)$.

For the following characteristic scales of synoptic motion in the planetary boundary layer at midlatitude,

$$U \sim 10\,ms^{-1},\ L \sim 1000\,km,\ T \sim 300K,\ \theta_o \sim 300K,\ \ \theta' \sim 4K;$$
$$N \sim 0.01s^{-1},\ J \sim 0.01J\ s^{-1}.$$

Equation (2.53)' can then be approximated by

$$\frac{\partial\theta'}{\partial t} + u\frac{\partial\theta'}{\partial x} + v\frac{\partial\theta'}{\partial y} + \left(\frac{N^2\theta_o}{g}\right)w \approx 0\,. \qquad (2.54)$$

Since $\theta'/\theta_o \approx T'/T_o$, the above equation can also be expressed as

$$\frac{\partial T'}{\partial t} + u\frac{\partial T'}{\partial x} + v\frac{\partial T'}{\partial y} + (\Gamma_d - \Gamma)w \approx 0\,, \qquad (2.55)$$

where Γ_d and Γ are dry and actual lapse rates, respectively.

[Supplementary reading]

(*From Lin 2007 – Mesoscale Dynamics, Cambridge University Press*)

7.3.2 Conditional instability

The method for determining static instability, as discussed above, assumes either no saturation or 100% saturation. However, the situation is significantly different if the air parcel becomes saturated as is lifted upward. In an unsaturated atmosphere, the *unsaturated moist Brunt-Vaisala frequency* (N_w) can be estimated using the following formula:

$$N_w^2 = \frac{g}{\bar{\theta}_v}\frac{\partial \bar{\theta}_v}{\partial z}, \tag{7.3.14}$$

where θ_v is the *virtual potential temperature* of the environmental air and is related to the *virtual temperature* by

$$\theta_v = T_v \left(\frac{p_s}{p}\right)^{R_d/c_p}, \tag{7.3.15}$$

where

$$T_v = \left(\frac{1+q/\varepsilon}{1+q}\right)T \approx \left(\frac{1+1.609q}{1+q}\right)T \approx (1+0.61q)T\,. \tag{7.3.16}$$

In the above equation, q is the water vapor mixing ratio and ε is the ratio of the molecular weight of water vapor (m_v) to that of dry air (m_d), which has a value of 0.622.

When an unsaturated air parcel is lifted upward, its temperature follows a dry adiabat path until it reaches the *lifting condensation level* (*LCL*) (Fig. 7.5). Further lifting will result in condensation and cause the temperature of the air parcel to follow a moist adiabat. The latent heat released from the condensation will warm the air parcel and slow down the moist adiabatic lapse rate (Γ_s), making it slower than the *dry lapse rate* (Γ_d). Suppose the value of the observed environmental lapse rate (γ) is between Γ_s and Γ_d (i.e. $\Gamma_s < \gamma < \Gamma_d$), and that the forcing is strong enough to lift the air parcel past its *LCL*. A continued lifting will then force the air parcel to cool at a rate of Γ_s, eventually reaching a level where the temperature of the air parcel and the environment are equal. Further lifting will cause the air parcel's temperature to surpass its environment temperature, causing it to accelerate upward due to the buoyancy force. Since the buoyancy force acts on the parcel during this step, no additional forcing is needed. In other words, the air parcel is now experiencing free convection. This level is known as the *level of free convection* (*LFC*). Since this type of air parcel instability is subjected to a finite-amplitude displacement from its initial level to the *LFC*, it is referred to as *conditional instability*. Thus, *the necessary conditions for conditional instability to occur are: (a)* $\Gamma_s < \gamma < \Gamma_d$ *and (b) a lifting of the air parcel past its LFC.*

The criterion for conditional instability can also be determined via the vertical gradient of the *saturation equivalent potential temperature* (θ_e^*) which is defined as the equivalent potential temperature of a hypothetically saturated atmosphere at the initial level. This hypothetical atmosphere has been set to mimic the thermal structure of the actual atmosphere. In other words, θ_e^* can be defined as the equivalent potential temperature that the air parcel would have if it were saturated initially at the same pressure and temperature, and can be calculated by

$$\theta_e^* = \theta e^{Lq_{vs}/c_pT} . \tag{7.3.17}$$

In order to derive the criterion for conditional instability, we consider an air parcel lifted from $z_o - \delta z$ to z_o. At $z_o - \delta z$, the air parcel is assumed to have the same potential temperature as that of the environment, $\bar{\theta} - (\partial\bar{\theta}/\partial z)\eta$, where $\bar{\theta}$ is the potential temperature of the environmental air at z_o, and $\eta = \delta z$ is the vertical displacement. The potential temperature of the air parcel experiences a change of $\delta\theta$ when it is lifted from $z_o - \eta$ to z_o, i.e. $[\bar{\theta} - (\partial\bar{\theta}/\partial z)\eta] + \delta\theta$. Thus, the buoyancy of the air parcel at z_o is

$$b = g\left(\frac{\theta - \bar{\theta}}{\bar{\theta}}\right) = -\frac{g}{\bar{\theta}}\frac{\partial\bar{\theta}}{\partial z}\eta + g\frac{\delta\theta}{\bar{\theta}} . \tag{7.3.18}$$

Substituting the heating rate ($\dot{q}$) from latent heat release, $\dot{q} = -L(Dq_{vs}/Dt)$, into (2.2.5) gives

$$\frac{\delta\theta}{\bar{\theta}} \approx -\delta\left(\frac{Lq_{vs}}{c_pT}\right) \approx -\frac{\partial}{\partial z}\left(\frac{Lq_{vs}}{c_pT}\right)\eta . \tag{7.3.19}$$

Substituting (7.3.19) into (7.3.18) and using the definition of θ_e^* leads to

$$b \approx -\left(\frac{g}{\bar{\theta}_e^*}\frac{\partial\bar{\theta}_e^*}{\partial z}\right)\eta . \tag{7.3.20}$$

Substituting the above equation into (7.3.2) for a moist atmosphere, gives us

$$\frac{D^2\eta}{Dt^2} + \left(\frac{g}{\bar{\theta}_e^*}\frac{\partial\bar{\theta}_e^*}{\partial z}\right)\eta = 0 . \tag{7.3.21}$$

Therefore, *the conditional stability criterion for a saturated layer of air becomes*

$$\frac{\partial\bar{\theta}_e^*}{\partial z}\begin{cases} > 0 & \text{conditionally stable} \\ = 0 & \text{conditionally neutral} \\ < 0 & \text{conditionally unstable} \end{cases} \tag{7.3.22}$$

Note that in addition to $\partial\bar{\theta}_e^*/\partial z < 0$, the release of conditional instability requires the air parcel to be lifted above its LFC. This requirement is not included in the above derivation (e.g. Sherwood 2000; Schultz et al. 2000). Parcel theory also neglects the effects of mass continuity and pressure perturbation (Xu 1986), as also known from dry static instability.

Figure 7.5 illustrates the concept of conditional instability, where an idealized sounding is plotted on a *skew-T log-p* thermodynamic diagram. The *LCL*, *LFC*, and

LNB for the air parcel originating at *A* are denoted in the figure. The amount of energy available for free convection is called the *convective available potential energy* (*CAPE*), which is defined as the work done by the buoyancy force in lifting an air parcel from its *LFC* to *LNB*,

$$CAPE = \int_{z_{LFC}}^{z_{LNB}} b\ dz = \int_{z_{LFC}}^{z_{LNB}} g\left(\frac{\bar{\rho}-\rho}{\rho}\right) dz = \int_{z_{LFC}}^{z_{LNB}} g\left(\frac{T-\bar{T}}{\bar{T}}\right) dz = \int_{z_{LFC}}^{z_{LNB}} g\left(\frac{\theta-\bar{\theta}}{\bar{\theta}}\right) dz \qquad (7.3.23)$$

In a thermodynamic diagram, *CAPE* is proportional to the area enclosed by the environmental temperature curve and the moist adiabat of the air parcel in between the *LFC* and *LNB* (Fig. 7.5). The moist adiabat follows a *saturated adiabatic process*, which assumes all of the condensates remain in the air parcel. The *moist adiabatic lapse rate* can be derived to be

$$\Gamma_s \equiv -\frac{dT}{dz} = \frac{\Gamma_d}{1+(L/c_p)(dq_{vs}/dT)}\,. \qquad (7.3.24)$$

The saturation water vapor mixing ratio is defined as the ratio of the mass of water vapor to the mass of dry air containing the vapor at saturation. *A saturated adiabatic process is almost identical to a pseudoadiabatic process*. This is because the heat carried by the condensates is negligible compared to that carried by the air parcel. The moist adiabatic lapse rate can be approximated by

$$\Gamma_s \equiv \frac{\Gamma_d[1+(Lq_{vs}/R_dT)]}{1+\varepsilon\, L^2 q_{vs}/(c_p R_d T^2)}\,, \qquad (7.3.25)$$

where $\varepsilon = m_v/m_d \approx 0.622$ as defined earlier. Observed values of Γ_s show that it is about 4 K km^{-1} near the ground in humid conditions, increases to 6 to 7 K km^{-1} in the middle troposphere and is nearly equal to the dry lapse rate of 9.8 K km^{-1} at high altitudes where the air is colder and holds less water vapor.

The total amount of potential energy for an air parcel lifted upward from a certain level z_i to its *LNB* can be calculated as follows:

$$CAPE_i = \int_{z_i}^{z_{LNB}} g\left(\frac{T-\bar{T}}{\bar{T}}\right) dz\,. \qquad (7.3.26)$$

The *CAPE* is also referred to as *static potential energy* or *available buoyant energy* and is represented as the *positive area* (*PA*) on a thermodynamic diagram if z_i is assumed to be at *LFC*. The positive area may then be defined as

$$PA = \int_{p_{LNB}}^{p_{LFC}} R_d (T-\bar{T})\, d(\ln p) = \int_{z_{LFC}}^{z_{LNB}} g\left(\frac{T-\bar{T}}{\bar{T}}\right) dz\,. \qquad (7.3.27)$$

On the other hand, the *negative area* (*NA*) on a thermodynamic diagram is the area confined by the dry adiabat (below *LCL*) or the moist adiabat (above *LCL*) to the left, and the sounding to the right, from the initial level to the *LFC* (Fig. 7.5). The negative area represents the energy needed to lift an air parcel vertically and dry adiabatically or pseudoadiabatically to its *LFC* and is also known as the *convective inhibition* (*CIN*). Mathematically, *CIN* is defined as

$$CIN = \int_{z_i}^{z_{LFC}} g\left(\frac{\bar{T}-T}{\bar{T}}\right) dz \,. \qquad (7.3.28)$$

In practice, the surface height (z_{sfc}) is used for z_i. It can be shown that $CAPE_i = PA - NA_i$, where $CAPE_i$ and NA_i are the CAPE and NA at z_i. Thus, *a positive* $CAPE_i$ *is a necessary condition for conditional instability to occur*, so that the air parcel has potential energy for convection. In the absence of horizontal advection, the maximum vertical velocity that can be realized by an air parcel occurs when all the potential energy is converted into kinetic energy, i.e. $w_{\max} = \sqrt{2CAPE}$, because $Dw/Dt \approx w\partial w/\partial z$.

When rain evaporates in sub-saturated air or when a solid precipitate (snow or hail) melts at the melting level, a downdraft will be generated by the cooled air. The maximum downdraft can be estimated as $-w_{\max} = \sqrt{2DCAPE_i}$, where $DCAPE_i$ is the *downdraft convective available potential energy* and is defined as

$$DCAPE_i = \int_{z_s}^{z_i} g\left(\frac{\bar{T}-T}{\bar{T}}\right) dz \,, \qquad (7.3.29)$$

where z_s is normally the surface or the level at which an air parcel descends from the initial level z_i, allowing a neutral buoyancy to be achieved.

Therefore, in addition to the commonly adopted lapse-rate definition of conditional instability, i.e. the environmental lapse rate lies between the dry and moist adiabatic lapse rates, an available-energy definition has also been proposed, i.e. an air parcel must possess positive buoyant energy (i.e., $CAPE_i > 0$). More precisely, for an unsaturated air parcel, the stability can be classified as: (a) No CAPE: stability for all vertical displacements, (b) CAPE > 0: instability for some finite vertical displacements, which contains two subcategories: (i) CAPE > CIN and (ii) CIN > CAPE (Schultz et al. 2000). The available-energy definition is more consistent with the concept of *subcritical instability* (Sherwood 2000), which is defined as an instability that requires a finite amplitude perturbation exceeding a critical amplitude (Drazin and Reid 1981). Thus, it has been suggested that the term "conditional instability" should be reserved only for the lapse-rate concept, and the term "*latent instability*" for the energy-based concept (Schultz et al. 2000).

In summary (also see Table 7.1), there exist six static stability states for dry and moist air:

(1) absolutely stable $\gamma < \Gamma_s$,

(2) saturated neutral $\gamma = \Gamma_s$,

(3) conditionally unstable $\Gamma_s < \gamma < \Gamma_d$,

(4) dry neutral $\gamma = \Gamma_d$,

(5) dry absolute unstable $\gamma > \Gamma_d$,

(6) moist absolutely unstable $\gamma_s > \Gamma_s$, (7.3.30)

where γ_s is the saturated lapse rate of the environmental air. Note that moist absolute instability is not equivalent to conditional instability. In a typical conditionally unstable situation, an initially unsaturated air parcel is lifted to saturation in an unsaturated environment. The air parcel will then follow a moist adiabat, and will become unstable after further lifting. However, under certain circumstances, an initially conditionally unstable atmosphere may become moist absolute unstable after lifting (Bryan and Fritsch 2000).

References

Holton, 2004: Introduction to Dynamic Meteorology. Elsvier.

Lin, Y.-L., 2007: *Mesoscale Dynamics*. Cambridge University Press, 630pp.

Bannon, P. R., 1986: Linear development of quasi-geostrophic baroclinic disturbances with condensational heating. J. Atmos. Sci., 43, 2261-2274.

Batchelor, G. K., 1953: The condition for dynamical similarity of motions of a frictionless perfect-gas atmosphere. Quart. J. Roy. Meteor. Soc., 79, 224-235.

Boussinesq, J., 1903: *Théorie analytique de la chaleur*, Vol. 2, Gauthier-Villars, Paris.

Durran, D. R., 1989: Improving the anelastic approximation. J. Atmos. Sci., 46, 1453-1461.

Eady, 1949: Long wave and cyclone waves. Tellus, 1, 33-52.

Emanuel, K., and D. J. Raymond, 1984: *Dynamics of Mesoscale Weather Systems*. Ed. J. B. Klemp, NCAR, 1984, 591pp.

Holton, J. R., 2004: *Introduction to Dynamic Meteorology*. 4th Ed., Elsevier Academic Press, Inc. 535pp.

Janowitz, G. S., 1977: The effects of compressibility on the stably stratified flow over a shallow topography in the beta plane. J. Atmos. Sci., 34, 1707-1714.

Nance, L. B., and D. R. Durran, 1994: A comparison of the accuracy of three anelastic systems and the pseudo-incompressible system. J. Atmos. Sci., 51, 3549-3565.

Ogura, Y., and N. A. Phillips, 1962: Scale analysis of deep and shallow convection in the atmosphere. J. Atmos. Sci., 19, 173-179.

Spiegel, E. A., and G. Veronis, 1960: On the Boussinesq approximation for a compressible fluid. Astro. Phys. J., 131, 442-447.