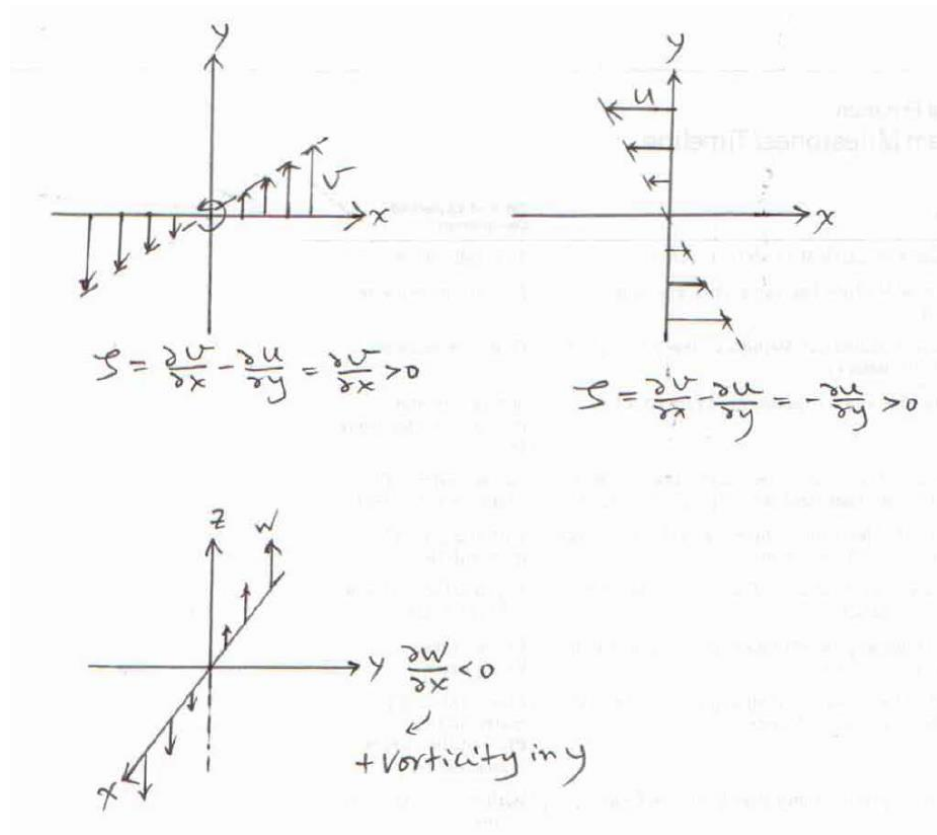


Chapter 5 Circulation and Vorticity

5.2 Vorticity

- Vorticity is a microscopic measure of rotation in a fluid.



- Definitions of **3D vorticity** and **absolute vorticity**

$$\boldsymbol{\omega} = \nabla_{\mathbf{x}} \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \mathbf{k},$$

$$\boldsymbol{\omega}_a = \nabla_{\mathbf{x}} \mathbf{V}_a = \boldsymbol{\omega} + \nabla_{\mathbf{x}} \mathbf{V}_e,$$

$$\zeta = \mathbf{k} \cdot \boldsymbol{\omega} = \mathbf{k} \cdot (\nabla_{\mathbf{x}} \mathbf{V}),$$

$$\zeta_a = \mathbf{k} \cdot \boldsymbol{\omega}_a = \mathbf{k} \cdot (\nabla_{\mathbf{x}} \mathbf{V}) + \mathbf{k} \cdot (\nabla_{\mathbf{x}} \mathbf{V}_e) = \zeta + f.$$

- Definitions of **vertical relative vorticity** (ζ) and **vertical absolute vorticity**:

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}; \quad \zeta_a = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + f.$$

- **Relation between ζ and C :**

Applying the Stokes' theorem to the definition of circulation, we may obtain the relation between vorticity and circulation:

$$C = \oint \mathbf{V} \cdot d\mathbf{l} = \iint_A (\nabla_{\mathbf{x}} \mathbf{V}) \cdot \mathbf{n} dA = \iint_A \zeta dA = \bar{\zeta} A \quad (4.8),$$

Stokes' Theorem: Stokes' theorem (e.g. see Adv. Calculus for appl. by Hildebrand) links contour integration to area integration,

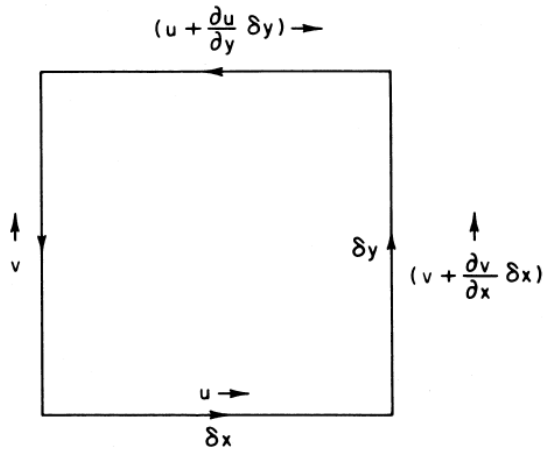
$$\oint \mathbf{V} \cdot d\mathbf{l} = \iint_A (\nabla_{\mathbf{x}} \mathbf{V}) \cdot \mathbf{n} dA,$$



where A is the surface area enclosed by the contour for contour integration and \mathbf{n} is a unit vector perpendicular to the surface in counterclockwise sense.

The above relation can also be obtained by evaluating the circulation along each side of a small rectangle:

$$\begin{aligned}\delta C &\equiv \oint \mathbf{V} \cdot d\mathbf{l} = u\delta x + \left(v + \frac{\partial v}{\partial x}\delta x\right)\delta y - \left(u + \frac{\partial u}{\partial y}\delta y\right)\delta x - v\delta y \\ &= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\delta x\delta y = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\delta A\end{aligned}$$



Thus, we have

$$\frac{\delta C}{\delta A} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

This implies that

$$C \equiv \oint \mathbf{V} \cdot d\mathbf{l} = \bar{\zeta} A \quad (4.8)''$$

In words, circulation is roughly equal to the mean vorticity times the area enclosed by the integration contour.

The above equation may also be rewritten as

$$\frac{DC}{DA} = \zeta \quad (4.8)$$

- **Vorticity in Natural Coordinates**

Definition of natural coordinates, (\mathbf{t}, \mathbf{n}) : \mathbf{t} is a unit vector tangential to the local velocity vector, \mathbf{n} is a unit vector perpendicular to \mathbf{t} pointing to the left.

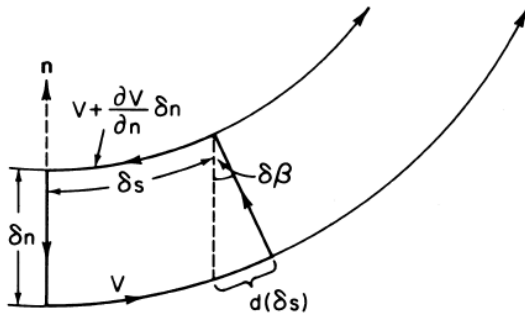


Fig. 4.5 Circulation for an infinitesimal loop in the natural coordinate system.

However, from Fig. 4.5, $d(\delta s) = \delta\beta \delta n$, where $\delta\beta$ is the angular change in the wind direction in the distance δs . Hence,

$$\delta C = \left(-\frac{\partial V}{\partial n} + V \frac{\delta\beta}{\delta s} \right) \delta n \delta s$$

or, in the limit $\delta n, \delta s \rightarrow 0$

$$\zeta = \lim_{\delta n, \delta s \rightarrow 0} \frac{\delta C}{(\delta n \delta s)} = -\frac{\partial V}{\partial n} + \frac{V}{R_s} \quad (4.9)$$

where R_s is the **radius of local curvature**.

In this coordinate system, the vertical vorticity is composed by the **curvature vorticity** (V/R) and **shear vorticity** ($-\partial V/\partial n$),

$$\zeta = \frac{V}{R} - \frac{\partial V}{\partial n},$$

where R is the radius of local curvature

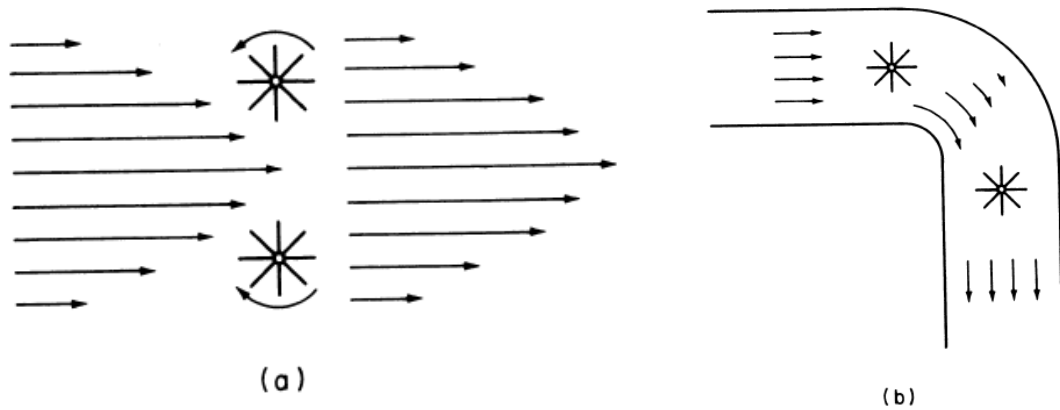


Fig. 4.6 Two types of 2-D flow with: (a) shear vorticity, and (b) curvature vorticity.