1.4 Alternate Vertical Coordinates

Any single-valued monotonic function, such as height or pressure, may be used as an independent vertical coordinate.

Traditionally, pressure has been used in synoptic observations, analysis, and NWP models as an alternate vertical coordinate (isobaric coordinate) due to the following reasons:

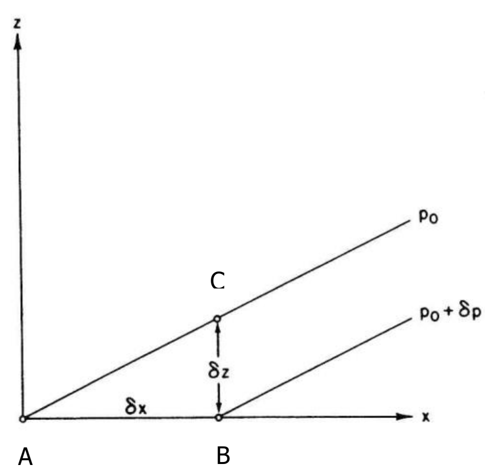
(1) Soundings give pressure directly.
(2) Continuity equation is in linear and time-independent form.
(3) Thermal wind and geostrophic wind relations are simpler.
(4) Density does not appear in the pressure gradient forces.

Isentropic coordinate ($\theta = \text{constant}$) has also been adopted for synoptic analysis, which is able to show more details of some weather systems, such as upper-level fronts.

(a) Derive a general formula for coordinate transformation

In the coordinate transformation, we need to express the horizontal pressure gradient force in the *x*-direction

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z$$



in isobaric (pressure) coordinate. When height (z) is used as a vertical coordinate, this horizontal derivative is evaluated by holding z constant. When we use pressure as a vertical coordinate, we now need to evaluate this derivative along a constant pressure surface.

In general, the pressure gradient on an *s*-surface can be derived as,

$$\left(\frac{\partial p}{\partial x}\right)_s = \lim_{\delta x \to 0}\left(\frac{\delta p}{\delta x}\right)_s \approx \frac{p_c - p_A}{\delta x} = \frac{p_c - p_B}{\delta x} + \frac{p_B - p_A}{\delta x} = \frac{p_c - p_B}{\delta z}\frac{\delta z}{\delta x} + \frac{p_B - p_A}{\delta x}$$
$$\approx \frac{\partial p}{\partial z}\left(\frac{\partial z}{\partial x}\right)_s + \left(\frac{\partial p}{\partial x}\right)_z ,$$

which gives

$$\left(\frac{\partial p}{\partial x}\right)_s = \frac{\partial p}{\partial z}\left(\frac{\partial z}{\partial x}\right)_s + \left(\frac{\partial p}{\partial x}\right)_z$$

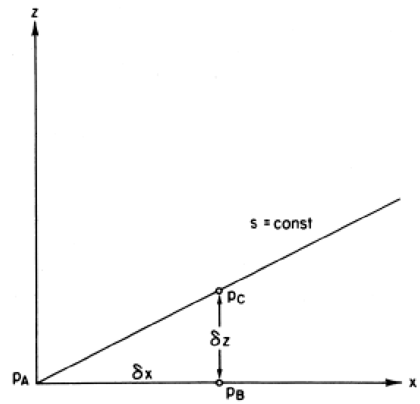


Fig. 1.12 Transformation of the pressure gradient force to *s* coordinates.

The above equation gives

$$0 = \left(\frac{\partial p}{\partial x}\right)_p = \frac{\partial p}{\partial z}\left(\frac{\partial z}{\partial x}\right)_p + \left(\frac{\partial p}{\partial x}\right)_z .$$

Therefore,

$$PGF = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -g\left(\frac{\partial z}{\partial x}\right)_p = -\left(\frac{\partial \phi}{\partial x}\right)_p,$$

where ϕ is the geopotential, defined as $\phi = \int_0^z g dz$, which is the work needed to lift an air parcel from surface to a height z.

(b) Vertical coordinates commonly adopted in meteorology

(1) Pressure (isobaric) coordinates ($s = p$)

As derived above,

$$PGF = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -g\left(\frac{\partial z}{\partial x}\right)_p = -\left(\frac{\partial \phi}{\partial x}\right)_p.$$

(2) Normalized pressure coordinates (σ-p)

"In simulating a mesoscale flow within a finite domain, the height vertical coordinate may propose problems. For example, it may intercept the terrain in a mountainous area and thus create problems in dealing with the lower boundary condition. Similar problems happen to the pressure coordinate and isentropic coordinate when isobaric surfaces and isentropic surfaces intercept the lower boundary, respectively, which may occur when there is strong orographic blocking. To avoid the problem, a vertical sigma coordinate, which matches the lowest coordinate surface with the bottom topography, has been proposed (Phillips 1957)."

[Quoted from Lin (2007) - Mesoscale Dynamics)]

$$s = \sigma = \frac{p(x,y,z,t)}{p_s(x,y,z,t)}.$$

It may be derived,

$$PGF = -\left(\frac{RT}{p_s}\right)\frac{\partial p_s}{\partial x} - \left(\frac{\partial \phi}{\partial x}\right)_\sigma .$$

(3) Isentropic coordinates

$$s = \theta = T\left(\frac{p_o}{p}\right)^{R_d/c_p} .$$

The pressure gradient force may be derived,

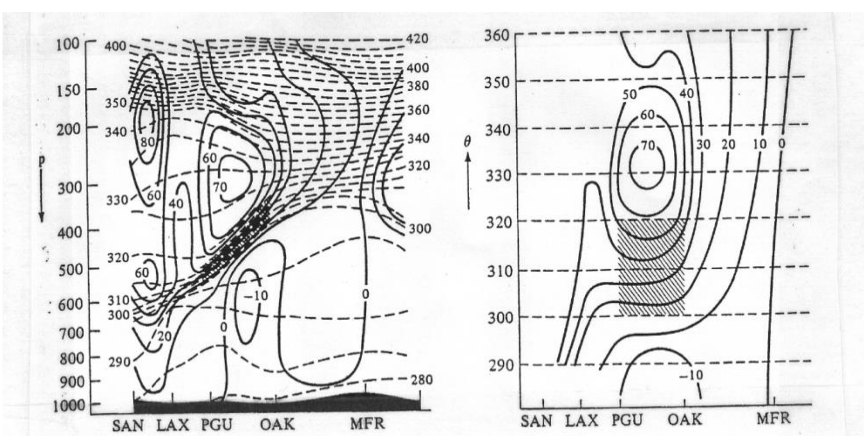
$$PGF = -\left[\frac{\partial}{\partial x}\left(c_p T + \phi\right)\right]_\theta .$$

where $c_p T + \phi$ is called the *Montgomery function*.

Advantages of using isentropic coordinate against p coordinate are:

(1) Showing some details of some weather systems, such as upper-level fronts.

(2) Making it easier to follow isentropic (adiabatic) motions.



isobaric coordinates isentropic coordinates

(4) Terrain-following coordinates (σ-z)

$$s = \sigma = \frac{z - z_s}{z_T - z_s},$$

where z_S and z_T are the terrain elevation and the top of model domain, respectively.

"When the sigma coordinate is applied to the height coordinates, it is called $\sigma - z$ or the terrain-following coordinates in which the σ can be defined as

$$\sigma = \frac{z_T(z - z_S)}{z_T - z_S}, \qquad (13.1.10)$$

where z_S is the height of the lower surface in the $\sigma - z$ coordinates, which is independent of time, and z_T is the constant domain height or the constant height of the terrain-following part of the domain. In the general σ coordinates, the pressure, p, can be written as

$$p(x, y, z, t) = p[x, y, \sigma(x, y, z, t), t]. \qquad (13.1.11)$$

The pressure gradient in x direction in the z coordinates can be obtained by performing the chain-rule,

$$\left(\frac{\partial p}{\partial x}\right)_z = \left(\frac{\partial p}{\partial x}\right)_\sigma + \frac{\partial p}{\partial \sigma}\left(\frac{\partial \sigma}{\partial x}\right)_z. \qquad (13.1.12)$$

The pressure gradient ($\partial p / \partial x$) in the σ coordinates can be obtained by deriving $(\partial \sigma / \partial x)_z$ from (13.1.10) and substituting it into (13.1.12):

$$\left(\frac{\partial p}{\partial x}\right)_\sigma = \left(\frac{\partial p}{\partial x}\right)_z - \left[\frac{\sigma - z_T}{z_T - z_S}\frac{\partial z_S}{\partial x}\right]\frac{\partial p}{\partial \sigma}. \qquad (13.1.13)$$

The sigma coordinate transformation may also be applied to the mass (hydrostatic pressure) coordinates (Skamarock et al. 2005), in addition to the $\sigma - z$ coordinates. If one replaces p by a general variable A, then the above transformation may be used to derive the gradient of A in x direction. One problem of the sigma vertical coordinate systems is that errors in two terms of the pressure gradient force do not cancel out (Smagorinski et al. 1967). To avoid this problem, the step-mountain or eta coordinates have been proposed (Mesinger et al. 1988)."

[Quoted from Lin (2007) - Mesoscale Dynamics)]

(5) ETA or step-mountain coordinates

(used in the ETA Model; Mesinger et al. 1988 MWR, Janjic' 1994 MWR)

In ETA coordinates, *eta* (η) is defined as

$$\eta = \frac{p - p_T}{p_S - p_T}\eta_S \qquad (13.1.14)$$

with

$$\eta_S = \frac{p_r(z_S) - p_T}{p_r(0) - p_T}. \qquad (13.1.15)$$

In the above equations, p is pressure; the subscripts T and S denote the top and surface values of the model atmosphere; z is geometric height, and $p_r(z)$ is a suitably defined reference pressure as a function of z. The boundary ... undary (η = 0), a coordinate is that it is of the first order of accuracy in representing the terrain, which is less accurate compared with the terrain-following coordinates, which is of the second-order of accuracy. A third method is to adopt a *finite-element scheme*, which approximates the mountain surface by one side of the finite elements. It has the same advantage as not having to transform the governing equations into complicated forms as well as having a higher-order accuracy compared to the step-mountain coordinates.

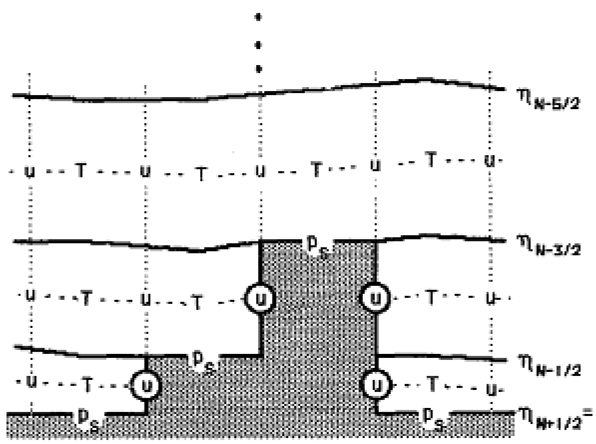


FIG. 1. A schematic picture of the representation of mountains using the coordinate (2.1)–(2.2).

(6) Flux-form equations in mass coordinates (used in the WRF model)

See http://wrf-model.org/PRESENTATIONS/2000_04_18_Klemp/index.htm (=>Numerics and Dynamic Solver)

Flux-Form Equations in Height Coordinates

Conservative variables: $U = \rho u, \quad V = \rho v, \quad W = \rho w, \quad \Theta = \rho\theta$

Inviscid, 2-D equations in Cartesian coordinates

$$\frac{\partial U}{\partial t} + \gamma R \pi \frac{\partial \Theta}{\partial x} - fV = -\frac{\partial Uu}{\partial x} - \frac{\partial Wu}{\partial z}$$

$$\frac{\partial W}{\partial t} + \gamma R \pi \frac{\partial \Theta}{\partial z} + g\rho = -\frac{\partial Uw}{\partial x} - \frac{\partial Ww}{\partial z}$$

$$\frac{\partial \Theta}{\partial t} + \frac{\partial U\theta}{\partial x} + \frac{\partial W\theta}{\partial z} = \rho Q$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial W}{\partial z} = 0$$

Pressure terms directly related to Θ:

$$\gamma R \pi \nabla\Theta = c_p \Theta \nabla \pi = \nabla p$$

Flux-Form Equations in Mass Coordinates

Hydrostatic pressure coordinate: $\eta = (\pi - \pi_t)/\mu, \qquad \mu = \pi_s - \pi_t$

Conservative variables: $U = \mu u, \quad W = \mu w, \quad \Theta = \mu\theta, \quad \Omega = \mu\dot{\eta}$

Inviscid, 2-D equations without rotation:

$$\frac{\partial U}{\partial t} + \mu\alpha \frac{\partial p}{\partial x} + \frac{\partial p}{\partial \eta}\frac{\partial \phi}{\partial x} = -\frac{\partial Uu}{\partial x} - \frac{\partial \Omega u}{\partial \eta}$$

$$\frac{\partial W}{\partial t} + g\left(\mu - \frac{\partial p}{\partial \eta}\right) = -\frac{\partial Uw}{\partial x} - \frac{\partial \Omega w}{\partial \eta}$$

$$\frac{\partial \Theta}{\partial t} + \frac{\partial U\theta}{\partial x} + \frac{\partial \Omega\theta}{\partial \eta} = \mu Q$$

$$\frac{\partial \mu}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial \Omega}{\partial \eta} = 0$$

$$\frac{d\phi}{dt} = gw, \qquad \frac{\partial \phi}{\partial \eta} = -\mu\alpha, \qquad p = \left(\frac{R\theta}{p_0\alpha}\right)^{\gamma}$$