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Ch.2 Scale Analysis and Application of the Basic Equations

(For equation editor: $DT/Dt = \partial T/\partial t + V \cdot \nabla T$)

From Sec. 3.1, we have derived the equation of motion

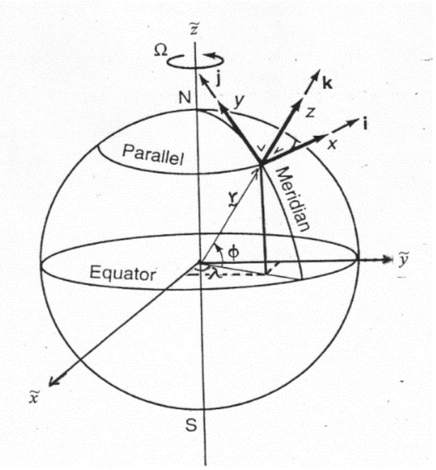
$$\frac{D\boldsymbol{V}}{Dt} = -\frac{1}{\rho}\nabla p - 2\boldsymbol{\Omega}\times\boldsymbol{V} + \boldsymbol{g} + \boldsymbol{F}_r\,, \qquad (2.8)$$

Coriolis force

where $\boldsymbol{g} = \boldsymbol{g}^* + \Omega^2\boldsymbol{R}$ is the effective gravity.

Equation (2.8) can be transformed into scalar components in spherical coordinates (Sec. 2.3, Holton)

- Coordinate systems on the Earth's Rotating Frame of Reference



Equations of motion on the local coordinates with earth curvature included:

$$\frac{Du}{Dt} = 2\Omega v \sin\phi - 2\Omega w \cos\phi - \frac{uw}{a} - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{uv\tan\phi}{a} + \nu\nabla^2 u \qquad (2.19)$$

$$\frac{Dv}{Dt} = -2\Omega u \sin\phi - \frac{vw}{a} - \frac{u^2\tan\phi}{a} - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v \qquad (2.20)$$

$$\frac{Dw}{Dt} = 2\Omega u \cos\phi + \frac{u^2 + v^2}{a} - \frac{1}{\rho}\frac{\partial p}{\partial z} - g + \nu\nabla^2 w \qquad (2.21)$$

2.1 Scale Analysis

Objectives of the scale analysis:

(1) To simplify the mathematics by eliminating insignificant terms in the equations, and

(2) To filter out the unwanted disturbances, such as sound waves and gravity waves in numerical weather prediction (NWP) simulations.

Definitions of atmospheric scales (Lin 2007 – Mesoscale Dynamics, Cambridge U. Press):

"Based on radar observations of storms, atmospheric motions can be categorized into the following three scales (Ligda 1951):

(a) ***synoptic (large) scale***: 1000 km < L

(b) ***mesoscale***: 20 km < L < 1000 km

(c) ***microscale***: L < 20 km

The atmospheric motions have also been categorized into 8 separate scales (Orlanski 1975; Table 1.1):

(a) *macroscale:* 2000 km < L < 10,000 km

macro-α (10,000 km < L) [planetary scale]

macro-β (2000 km < L < 10 000 km)

[synoptic scale to planetary scale]

(b) ***mesoscale:*** **2 km < *L* < 2000 km**

meso-α (200 km < L < 2000 km)

meso-β (20 km < L < 200 km)

meso-γ (2 km < L < 20 km)

(c) ***microscale:*** **2 m < *L* < 2 km**

micro-α (200 m < L < 2 km)

micro-β (20 m < L < 200 m)

micro-γ (2 m < L < 20 m) scales

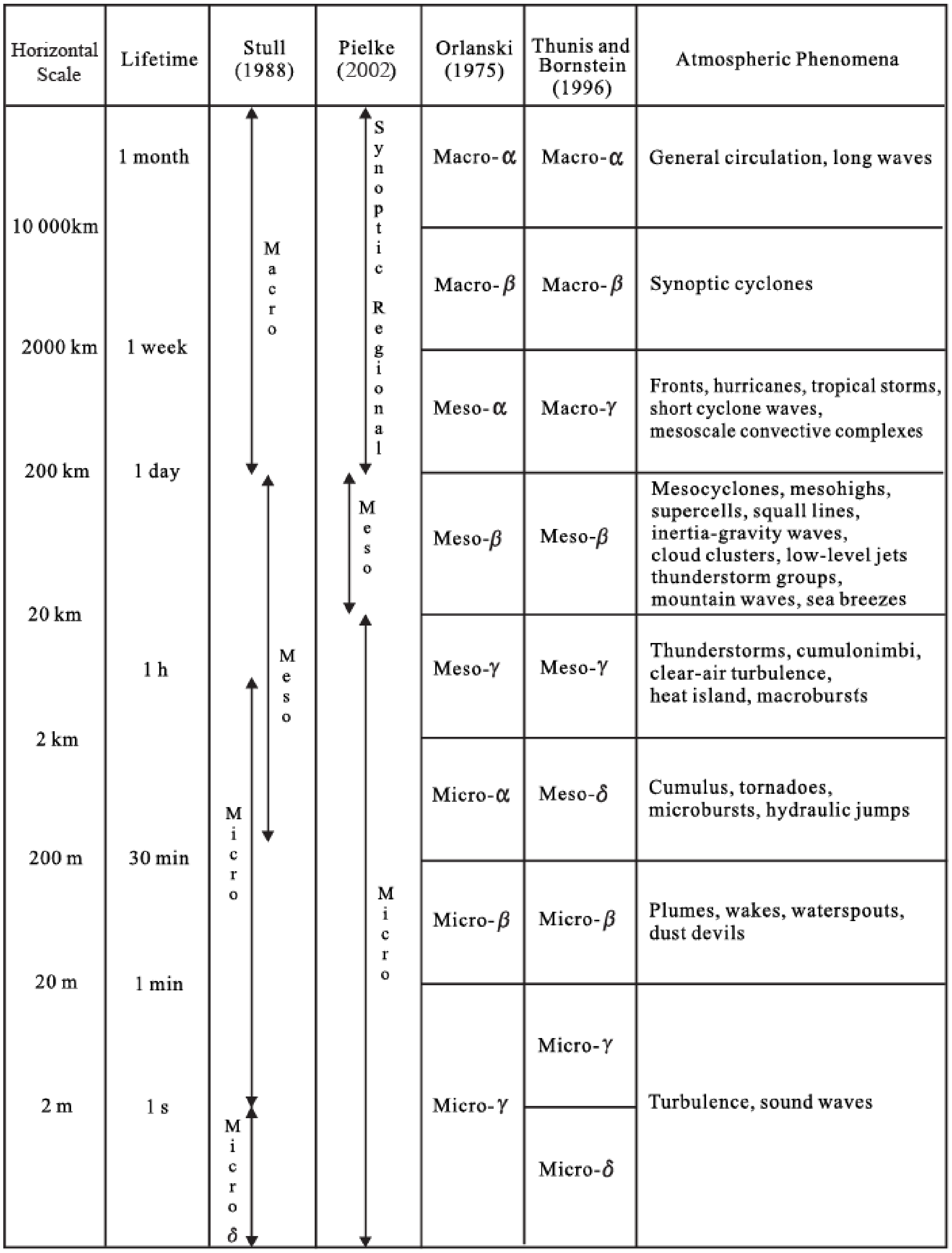
Based on theoretical considerations, the following different scales for atmospheric motions can be defined (Emanuel and Raymond 1984):

(a) ***synoptic (large or macro) scale***: for motions which are quasi-geostrophic and hydrostatic,

(b) ***mesoscale***: for motions which are non-quasi-geostrophic and hydrostatic, and

(c) ***microscale***: for motions which are non-geostrophic, nonhydrostatic, and turbulent

Table 1.1 Atmospheric scale definitions. (Adapted after Thunis and Borstein 1996)
(From Lin 2007 – Mesoscale Dynamics, Cambridge U. Press)



Horizontal Scale	Lifetime	Stull (1988)	Pielke (2002)	Orlanski (1975)	Thunis and Bornstein (1996)	Atmospheric Phenomena
10 000km	1 month	Macro	Synoptic Regional	Macro-α	Macro-α	General circulation, long waves
2000 km	1 week			Macro-β	Macro-β	Synoptic cyclones
200 km	1 day			Meso-α	Macro-γ	Fronts, hurricanes, tropical storms, short cyclone waves, mesoscale convective complexes
20 km		Meso	Meso	Meso-β	Meso-β	Mesocyclones, mesohighs, supercells, squall lines, inertia-gravity waves, cloud clusters, low-level jets thunderstorm groups, mountain waves, sea breezes
2 km	1 h	Micro	Micro	Meso-γ	Meso-γ	Thunderstorms, cumulonimbi, clear-air turbulence, heat island, macrobursts
200 m	30 min			Micro-α	Meso-δ	Cumulus, tornadoes, microbursts, hydraulic jumps
20 m	1 min			Micro-β	Micro-β	Plumes, wakes, waterspouts, dust devils
				Micro-γ	Micro-γ	Turbulence, sound waves
2 m	1 s	Micro δ			Micro-δ	

(a) Horizontal momentum equation

The vector form of the momentum equation in the rotating frame of reference

$$\frac{D\boldsymbol{V}}{Dt} = -\frac{1}{\rho}\nabla p - 2\boldsymbol{\Omega}\times\boldsymbol{V} + \boldsymbol{g} + \boldsymbol{F}_r\,, \qquad (2.8)$$

can be transformed into scalar components in spherical coordinates (Sec. 2.3, Holton)

$$\frac{Du}{Dt} = 2\Omega v\sin\phi - 2\Omega w\cos\phi - \frac{uw}{a} - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{uv\tan\phi}{a} + \nu\nabla^2 u \qquad (2.19)$$

$$\frac{Dv}{Dt} = -2\Omega u\sin\phi - \frac{vw}{a} - \frac{u^2\tan\phi}{a} - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v \qquad (2.20)$$

$$\frac{Dw}{Dt} = 2\Omega u\cos\phi + \frac{u^2+v^2}{a} - \frac{1}{\rho}\frac{\partial p}{\partial z} - g + \nu\nabla^2 w \qquad (2.21)$$

In this course, we will focus on midlatitude weather systems which have the following characteristic scales:

$U \sim 10\ ms^{-1}$	horizontal velocity scale
$W \sim 1\ cm\ s^{-1}$ or $10^{-2}\ ms^{-1}$	vertical velocity scale
$L \sim 1000\ km$ or $10^6\ m$	horizontal length scale
$L_z \sim 10\ km$ or $10^4\ m$	vertical length scale
$(\delta p)_{x,y} \sim 10\ mb$ or $10^3\ Pa$	horizontal pressure perturb. scale
$T \sim L/U = 10^5\ s$	time scale
$\rho_o \sim 1\ kg\ m^{-3}$	density scale
$f_o \sim 10^{-4}\ s^{-1}$	Coriolis parameter ($\sim 2\Omega \sin 45^o$)
$a \sim 10^7\ m$ ($\sim$6400 km)	Earth radius
$\nu \sim 10^{-5}\ m^2 s^{-1}$	coefficient of molecular friction

Scale analysis of the horizontal momentum equations:

$$\frac{Du}{Dt}-2\Omega v\sin\phi+2\Omega w\cos\phi+\frac{uw}{a}-\frac{uv\tan\phi}{a}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u \quad (2.19)$$

Scales	$\frac{U^2}{L}\left(\frac{UW}{L_z}\right)$	f_oU	f_oW	$\frac{UW}{a}$	$\frac{U^2}{a}$	$\frac{\delta p}{\rho_o L}$	$\frac{\nu U}{L^2}\left(\frac{\nu W}{L_z^2}\right)$
Magnitude (in m/s^2)	$10^{-4}\left(10^{-5}\right)$	10^{-3}	10^{-6}	10^{-8}	10^{-5}	10^{-3}	$10^{-16}\ (10^{-15})$

(1) Geostrophic approximation

Keeping the terms with highest order of magnitude (10^{-3}) gives the geostrophic wind

$$-fv_g=-\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad (2.22a)$$

where $f=2\Omega\sin\phi$ is called the "Coriolis parameter" and v_g is called the geostrophic wind.

Similarly, the geostrophic wind in y direction can be derived

$$fu_g=-\frac{1}{\rho}\frac{\partial p}{\partial y}. \qquad (2.22b)$$

Equations (2.22a) and (2.22b) can be written in vector form

$$f\boldsymbol{V}_g=\boldsymbol{k}\,\mathrm{x}\,\frac{1}{\rho}\nabla p \qquad (2.23)$$

where $\boldsymbol{V}_g=u_g\boldsymbol{i}+v_g\boldsymbol{j}$ is the geostrophic wind velocity.

Characteristics of the geostrophic wind:

(a) V_g approximates the actual wind to within 10 – 15%

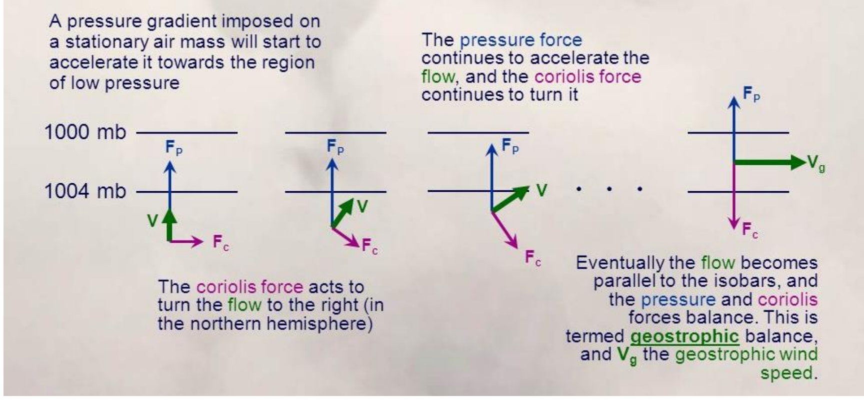
(b) V_g // isobars leaving the low to the left in Northern Hemisphere.

(c) V_g is larger at smaller spacing of the isobars.

(d) V_g is time independent.

- Examples of geostrophic adjustment problem

(a)



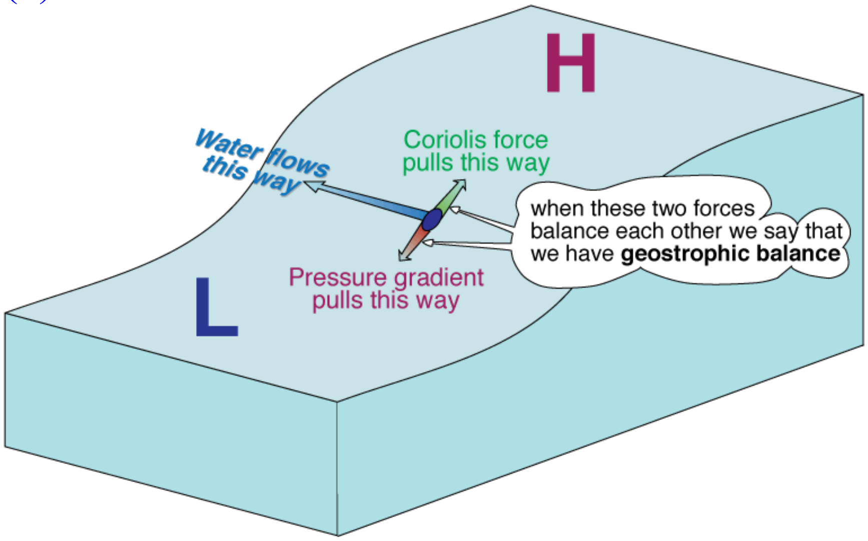
A pressure gradient imposed on a stationary air mass will start to accelerate it towards the region of low pressure

The pressure force continues to accelerate the flow, and the coriolis force continues to turn it

The coriolis force acts to turn the flow to the right (in the northern hemisphere)

Eventually the flow becomes parallel to the isobars, and the pressure and coriolis forces balance. This is termed **geostrophic** balance, and V_g the geostrophic wind speed.

(b)



when these two forces balance each other we say that we have **geostrophic balance**

(2) Approximate prognostic equations (see additional note)

If we keep all terms of O(10^{-4}) and higher, then we have

$$\frac{Du}{Dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{2.24}$$

$$\frac{Dv}{Dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y}\,. \tag{2.25}$$

Inertial Force   Coriolis Force   PGF

Equations (2.24) and (2.25) reduce to (2.22a) and (2.22b), respectively, whenever the first terms on the left hand side are very small compared to other terms, e.g.

$$\frac{Inertial\ \ Force}{Coriolis\ \ Force} = \frac{Du/Dt}{fv} \approx \frac{U/T}{fU} = \frac{1}{fT} << 1$$

where *T* is the time scale.

In an Eulerian frame of reference, the time scale can be calculated by *T* = *L*/*U*. Substituting it into the above equation leads to

$$R_o \equiv \frac{U}{fL} << 1$$

R_o is called the Rossby number.

Considering an air parcel following the motion, a *Lagrangian Rossby number* may be defined as

$$R_o = \frac{1}{fT} = \frac{1}{f(2\pi R/V_T)} = \frac{V_T}{2\pi fR}\,,$$

where *R* is the radius of a circular motion or radius of local curvature. Sometimes the Lagrangian Rossby number is defined as

$$R_o = \frac{\omega}{f} = \frac{2\pi}{fT} = \frac{V_T}{fR}\,.$$

(From Lin 2007) [Note that when $\omega \equiv 2\pi/T$, ω is called angular frequency, i.e. the frequency is measured by angle, instead of by cycle.]

Phenomenon	**Time scale**	**Lagrangian R_o** ($\approx \omega / f = 2\pi / fT$)
Tropical cyclone	$2\pi R / V_T$	V_T / fR
Inertia-gravity waves	$2\pi / N$ to $2\pi / f$	N / f to 1
Sea/land breezes	$2\pi / f$	1
Cumulus clouds	$2\pi / N_w$	N_w / f
Kelvin-Helmholtz waves	$2\pi / N$	N / f
PBL turbulence $2\pi h / U^*$	U^* / fh	
Tornadoes	$2\pi R / V_T$	V_T / fR

where

R = radius of maximum wind scale
ω = frequency
T = time scale
V_T = maximum tangential wind scale
f = Coriolis parameter
N = buoyancy (Brunt-Vaisala) frequency
N_w = moist buoyancy (Brunt-Vaisala) frequency
U^* = scale for friction velocity
h = scale for the depth of planetary boundary layer.

Comparing the inertial force to the viscous force leads to the definition of the Reynolds number (homework problem).

(b) Vertical momentum equation (see additional note)

$$\frac{Dw}{Dt} - 2\Omega u \cos\phi - \frac{u^2 + v^2}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \nu \nabla^2 w$$

Scales	$\frac{UW}{L}\left(or\,\frac{W^2}{L_z}\right)$	$f_o U$	$\frac{U^2}{a}$	$\frac{\delta p_z}{\rho_o H}$	g	$\frac{\nu W}{L_z^2}\left(or\,\frac{\nu W}{L^2}\right)$
Magnitude (in m/s^2)	$10^{-7}\ \left(10^{-8}\right)$	10^{-3}	10^{-5}	10	10	$10^{-15} or 10^{-19}$

From basic atmospheric structure

Keeping the terms of largest order of magnitude leads to:

(1) Hydrostatic equation

$$-\frac{1}{\rho}\frac{\partial p}{\partial z}-g=0$$

Under this approximation, the gravitational force is balanced by the vertical PGF approximately. The approximation is called "hydrostatic approximation".

Note that it is misleading to merely show the vertical acceleration (*Dw/Dt*) term is much smaller than the vertical PGF term. It is necessary to compare it to the perturbation PGF.

$$\frac{\partial p'}{\partial z}=-\rho' g\,. \tag{2.28}$$

[Reading assignment] Holton's p. 41-42 for details.

(2) Important terminologies

- Geopotential
- Geopotential height
- Hypsometric equation
- Scale height

(a) Geopotential

As discussed earlier, geopotential is defined as the work done when an air parcel of unit mass (1 kg) is lifted from sea level to a certain height *z*

(AMS Glossary of Meteorology: http://amsglossary.allenpress.com/glossary/search?id=geopotential-height1)

$$\phi = \int_0^z g dz$$

(b) Geopotential height

Geopotential height Z is defined as the height of a given point in the atmosphere in units proportional to the potential energy of unit mass (geopotential) at this height relative to sea level.

$$Z = \frac{1}{g_o} \int_0^z g \ dz \ .$$

- The actual height of an air parcel and the geopotential height are numerically interchangeable for most meteorological purposes.
- Higher (Lower) Z ⇔ higher (lower) pressure (give example here)

(c) Hypsometric equation

Integrating the equation of geopotential definition from z_1 to z_2 and substituting the hydrostatic equation into it leads to (reading assignment)

$$\phi(z_2) - \phi(z_1) = g_o(Z_2 - Z_1) = R\int_{p_1}^{p_2} T \ d(\ln p) . \qquad (1.21)$$

Equation (1.21) can be approximated by

$$\phi(z_2) - \phi(z_1) = g_o(Z_2 - Z_1) = -R\overline{T} \ln \frac{p_1}{p_2}$$

Thus, the physical meaning of the hypsometric equation is that the depth of an atmospheric layer is proportional to the mean layer temperature.

(d) Scale height

The height where the sea-level density (ρ_o) is reduced to its e-folding value ($\rho_o e^{-1}$). Note that approximately the air density is reduced exponentially

$$\rho(z) = \rho_o e^{-z/H} .$$

Thus, $z = H$ is the scale height.