ASME 434 Atmospheric Dynamics II Dr. Yuh-Lang Lin
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Lecture Note 4 The Vorticity Equation

(Equation editor: $\theta_o + \delta\theta$)

- Derivation of the vorticity equation

Cross-differentiating the zonal and meridional component equations with respect to *x* and *y* gives:

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}\right) \tag{4.10}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}\right) \tag{4.11}$$

Subtracting (4.10) from (4.11) and recalling that $\zeta = \partial v/\partial x - \partial u/\partial y$ lead to the vorticity equation:

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z} + (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + v\frac{\partial f}{\partial y} = \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) \tag{4.12}$$

or

$$\frac{\partial \zeta}{\partial t} = -\boldsymbol{V}\cdot\nabla(\zeta + f) - (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)$$
$$+ v\frac{\partial f}{\partial y} + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

Using $\frac{Df}{Dt} = v\frac{\partial f}{\partial y}$, (4.16) can also be rewritten in the form

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) \qquad (4.13)$$

- Physical meaning of individual terms of the vorticity equation Eq. (4.17) may also be written as

$$\frac{\partial \zeta}{\partial t} = -\boldsymbol{V}\cdot\nabla(\zeta + f) - (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

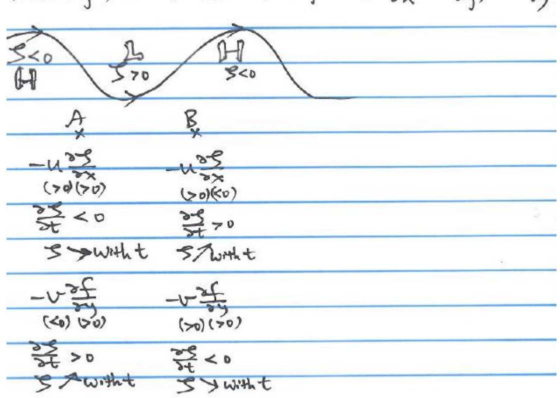
Local rate of change of vertical vorticity — Vorticity advection — Divergence (stretching) term

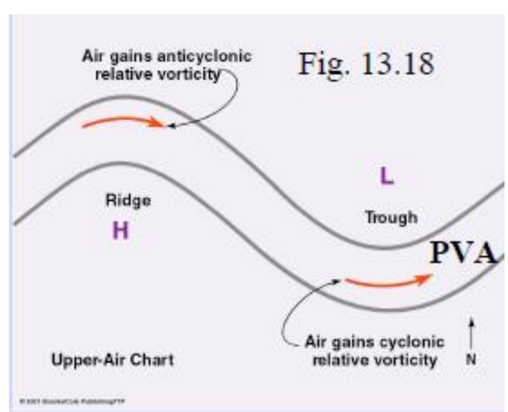
$$-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

Tilting (twisting) term — Solenoidal term

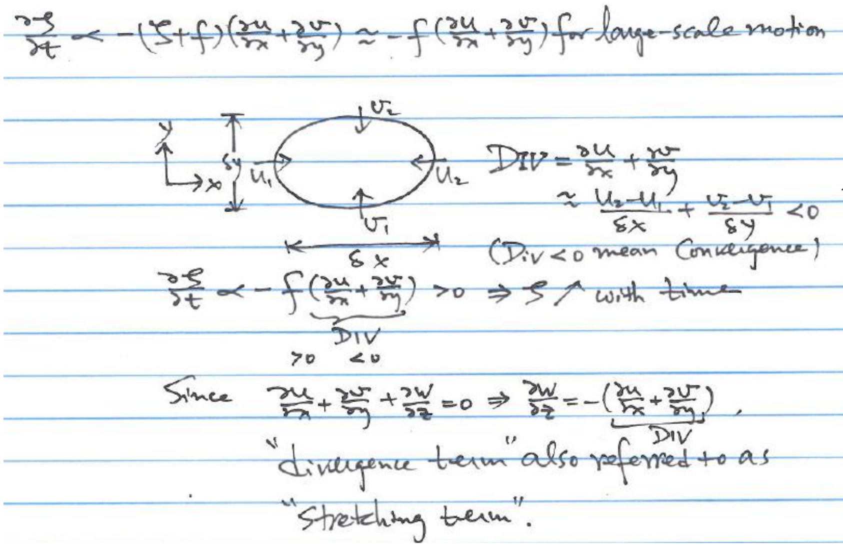
- Physical interpretation of the vorticity advection term

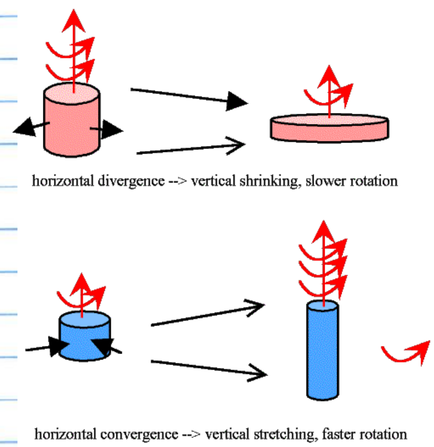
-V. $\nabla(\zeta+f) = -\vec{V}\cdot\nabla\zeta - \vec{V}\cdot\nabla f = -(u\frac{\partial\zeta}{\partial x}+v\frac{\partial\zeta}{\partial y}) - v\frac{\partial f}{\partial y}$



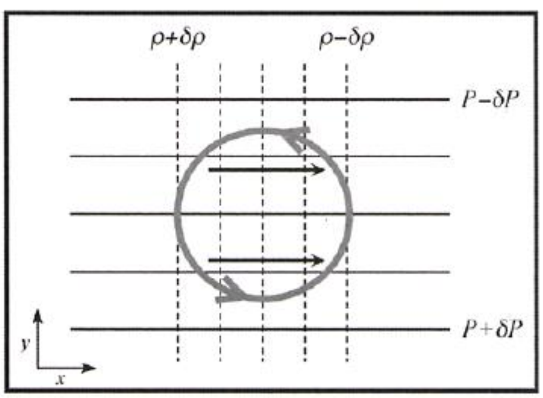


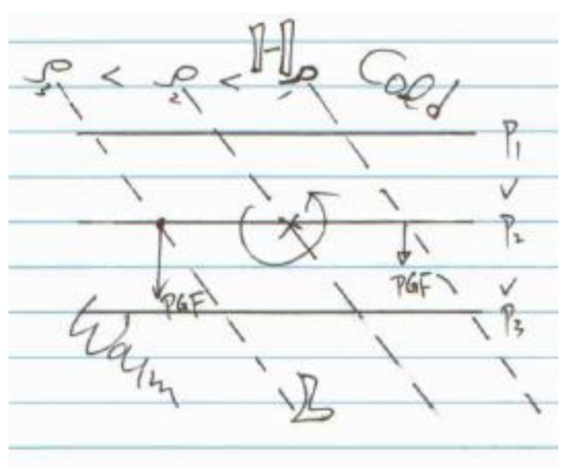
- Physical interpretation of the divergence (stretching) term





- Physical interpretation of the solenoidal term (also see the section of sea-breeze circulation)





$$\frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial\rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

The solenoidal term may also be expressed as

$$-\left(\frac{\partial\alpha}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial\alpha}{\partial y}\frac{\partial p}{\partial x}\right)\sim\left(\nabla p \mathrm{x} \nabla T\right)\cdot\boldsymbol{k}$$

- Physical interpretation of the tilting term

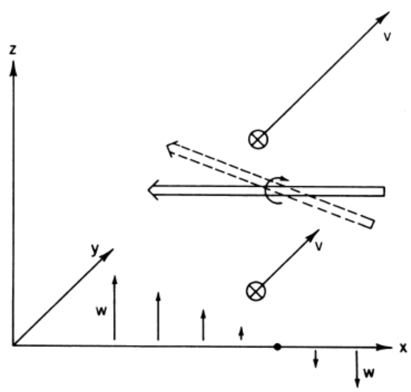


Fig. 4.12 Vorticity generation by the tilting of a horizontal vorticity vector (double arrow).

Application to the formation of mesocyclones and tornadoes

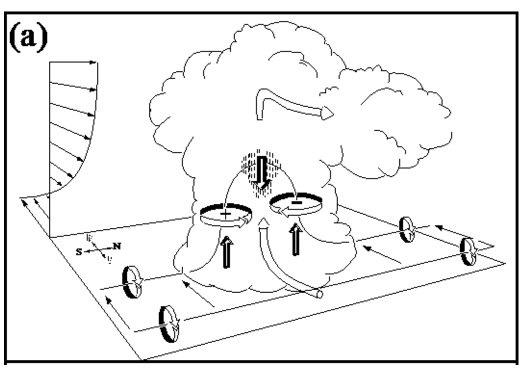


Fig. 8.20: A schematic depicting rotation development and the storm splitting. (a) Rotation development: In the early stage, a pair of vortices forms through tilting of horizontal vorticity associated with the (westerly) environmental shear. (b) Storm splitting: In the later stage, the precipitation induced downdraft splits the updraft. At this stage, vortex lines are tilted downward, producing two vortex pairs. Cylindrical arrows denote the direction of the storm-relative airflow, and heavy solid lines represent vortex lines with the sense of rotation denoted by circular arrows. Shaded arrows represent the forcing promoting new updraft and downdraft

acceleration. Vertical dashed lines denote regions of precipitation. Frontal symbols at the surface mark the boundary of cold air outflow. (Lin 2007, After Klemp)

- Vorticity Equation in Isobaric (pressure) Coordinates

$$\frac{\partial \zeta}{\partial t} = -\boldsymbol{V} \cdot \nabla(\zeta + f) - \omega \frac{\partial \zeta}{\partial p} - (\zeta + f)\nabla \cdot \boldsymbol{V} + k \cdot \left(\frac{\partial \boldsymbol{V}}{\partial p} \text{ x } \nabla \omega \right) \quad (4.21)$$

Note the solenoidal term is implicit.

- Scale analysis of the Vorticity Equation

Based on typical observed magnitudes for synoptic-scale motions, scales are chosen as follows:

$U \sim 10\,\text{m s}^{-1}$	horizontal scale
$W \sim 1\,\text{cm s}^{-1}$	vertical scale
$L \sim 10^6\,\text{m}$	length scale
$H \sim 10^4\,\text{m}$	depth scale
$\delta p \sim 10\,\text{hPa}$	horizontal pressure scale
$\rho \sim 1\,\text{kg m}^{-3}$	mean density
$\delta\rho/\rho \sim 10^{-2}$	fractional density fluctuation
$L/U \sim 10^5\,\text{s}$	time scale
$f_0 \sim 10^{-4}\,\text{s}^{-1}$	Coriolis parameter
$\beta \sim 10^{-11}\,\text{m}^{-1}\,\text{s}^{-1}$	"beta" parameter

The vorticity equation may be approximated by

$$\frac{D_h(\zeta + f)}{Dt} = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \quad (4.22b)$$

Where $D_h / Dt \equiv \partial / \partial t + u\partial / \partial x + v\partial / \partial y$.