

1.3 Apparent (Virtual) Forces and Coordinate Systems

Objective: To apply Newton's 2nd Law in a non-inertial frame of reference associated with Earth's rotation.

In order to do so, two additional apparent or virtual forces are emerged from the derivations: [centrifugal force](#) and [Coriolis force](#).

(a) Total differentiation of a scalar

Consider releasing a balloon at a place with

$T = T_o(x_o, y_o, z_o, t_o)$ to another place with

$T + \delta T = T(x_o + \delta x, y_o + \delta y, z_o + \delta z, t_o + \delta t)$, the temperature change following the balloon may be derived by applying the [chain rule](#),

$$\delta T = \left(\frac{\partial T}{\partial t} \right) \delta t + \left(\frac{\partial T}{\partial x} \right) \delta x + \left(\frac{\partial T}{\partial y} \right) \delta y + \left(\frac{\partial T}{\partial z} \right) \delta z .$$

Dividing the above equation by δt and taking $\lim_{\delta T \rightarrow 0}$ lead to

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u \left(\frac{\partial T}{\partial x} \right) + v \left(\frac{\partial T}{\partial y} \right) + w \left(\frac{\partial T}{\partial z} \right) \quad \text{or}$$
$$\frac{\partial T}{\partial t} = \frac{DT}{Dt} - \mathbf{V} \cdot \nabla T .$$

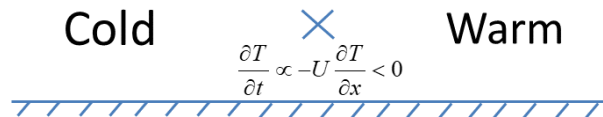
Physical meaning:

$\frac{DT}{Dt}$: Total rate of change of temperature following the motion

$\frac{\partial T}{\partial t}$: Local rate of change of temperature at a fixed location

$-V \cdot \nabla T$: Temperature advection

$$\frac{\partial T}{\partial t} = \frac{DT}{Dt} - U \frac{\partial T}{\partial x}$$



Example: **Cold advection** associated with a flow from cold to warm region.

(b) Total differentiation of a vector in a rotating system

For any vector \mathbf{A} , decompose it into three components in an inertial frame of reference or a rotating frame of reference (e.g., on a merry-go-around),

$$\mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k} = A'_x \mathbf{i}' + A'_y \mathbf{j}' + A'_z \mathbf{k}'$$

(inertial frame) (rotating frame)

Taking the total derivative of \mathbf{A} gives

$$\frac{D_a \mathbf{A}}{Dt} = \frac{DA_x}{Dt} \mathbf{i} + \frac{DA_y}{Dt} \mathbf{j} + \frac{DA_z}{Dt} \mathbf{k}$$

(Note $\frac{D_a A_x}{Dt} = \frac{DA_x}{Dt}$ for scalars)

$$= \frac{DA'_x}{Dt} \mathbf{i}' + \frac{DA'_y}{Dt} \mathbf{j}' + \frac{DA'_z}{Dt} \mathbf{k}' + A'_x \frac{D\mathbf{i}'}{Dt} + A'_y \frac{D\mathbf{j}'}{Dt} + A'_z \frac{D\mathbf{k}'}{Dt}$$

$$= \frac{DA}{Dt} + \left(A'_x \frac{D\mathbf{i}'}{Dt} + A'_y \frac{D\mathbf{j}'}{Dt} + A'_z \frac{D\mathbf{k}'}{Dt} \right)$$

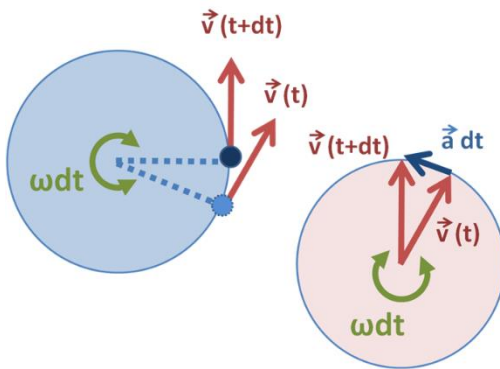


Total derivative of \mathbf{A} in the rotating frame

The above equation leads to

$$\frac{D_a \mathbf{A}}{Dt} = \frac{DA}{Dt} + \boldsymbol{\Omega} \times \mathbf{A}, \tag{2.2}$$

where $\boldsymbol{\Omega}$ is the rotation vector. (Example: Think about [centripetal force](#) in a circular motion. See below, but let $\boldsymbol{\Omega}$ be $\boldsymbol{\omega}$.)



Applying (2.2) to a position vector \mathbf{r} leads to

$$\frac{D_a \mathbf{r}}{Dt} = \frac{D\mathbf{r}}{Dt} + \boldsymbol{\Omega} \times \mathbf{r} \quad \text{or} \quad \mathbf{V}_a = \mathbf{V} + \boldsymbol{\Omega} \times \mathbf{r}. \quad (2.5)$$

Applying (2.2) to (2.5) again leads to

$$\frac{D_a \mathbf{V}_a}{Dt} = \frac{D\mathbf{V}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{V} - \Omega^2 \mathbf{R} \quad (2.7)$$

Note that Newton's second law states

$$\mathbf{a} = \frac{D_a \mathbf{V}_a}{Dt} = \sum_i \frac{\mathbf{F}_i}{m} \quad (2.3)$$

where \mathbf{F}_i 's are real forces.

Combining (2.3), real force expressions, and (2.7) gives the equation of motion in the rotating frame of reference,

$$\frac{D_a \mathbf{V}_a}{Dt} = \frac{D\mathbf{V}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{V} - \Omega^2 \mathbf{R} = -\frac{1}{\rho} \nabla p + \mathbf{g}^* + \mathbf{F}_r$$

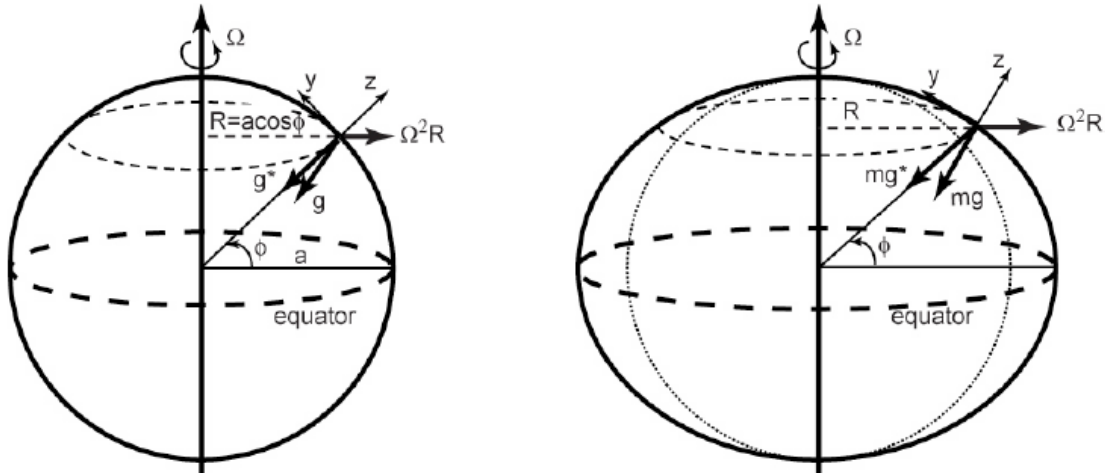
Centripetal force (real)

or

$$\frac{D\mathbf{V}}{Dt} = -\frac{1}{\rho} \nabla p - 2\boldsymbol{\Omega} \times \mathbf{V} + \mathbf{g} + \mathbf{F}_r, \quad (2.8)$$

Coriolis force Centrifugal force (apparent)

where $\mathbf{g} = \mathbf{g}^* + \Omega^2 \mathbf{R}$ is the effective gravity.



(c) Coordinate systems to be considered

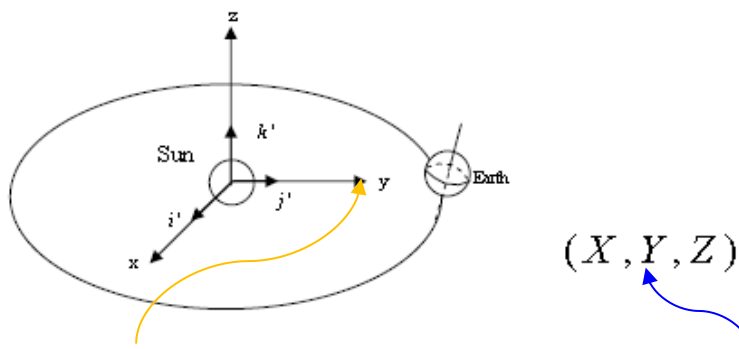
An **inertial (absolute) frame of reference** is a space-time coordinate system that neither rotates nor accelerates. In real life, such a frame of reference is purely theoretical, because gravitational force (and thus acceleration) exists everywhere in the known universe. However, they may be approximated very well in intergalactic space, or to a lesser extent within the confines of a coasting spacecraft.

- For convenience, let us assume that the solar system is an inertial frame of reference (which, in fact, is moving at a speed of ~ 250 km/s around the Milky Way Galaxy – see figure below)

(<http://www.enchantedlearning.com/subjects/astronomy/planets/earth/Speeds.shtml>)

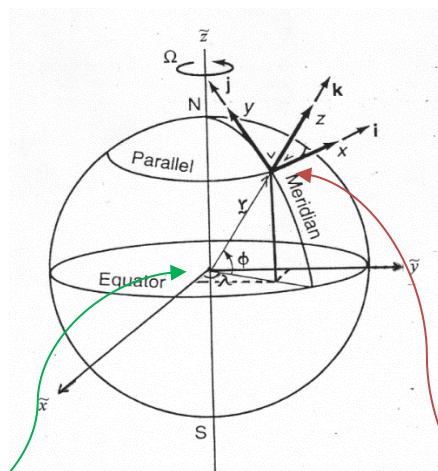


- The Earth's Rotating Frame of Reference



Inertial frame of reference (x, y, z)

Earth rotating frame of reference



Earth spherical coordinates (λ, ϕ, r)

Local coordinates (x, y, z)