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Chapter 4 Elementary Applications of the Basic Equations

4.1 Basic Equations in Isobaric Coordinates

(Ref.: Holton Sec. 3.1)

• The Horizontal Momentum Equation

The approximate horizontal momentum equations (2.24) and (2.25)

$$\frac{Du}{Dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (2.24)$$

$$\frac{Dv}{Dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y}\,. \qquad (2.25)$$

Inertial Force Coriolis Force PGF

may be written in the isobaric (pressure) coordinates by replacing the following PGF

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial \phi}{\partial x}\right)_p; \quad -\frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right)_z = -\left(\frac{\partial \phi}{\partial y}\right)_p \qquad (1.20)$$

into them

$$\frac{Du}{Dt} = fv - \frac{\partial \phi}{\partial x}, \quad \frac{Dv}{Dt} = -fu - \frac{\partial \phi}{\partial y} \qquad (3.2a\text{-}b)$$

Equations (2.24)-(2.25) and (3.2a)-(3.2b) can be written in vectorial form as

$$\frac{D\boldsymbol{V}}{Dt} + f\,\boldsymbol{k}\,x\boldsymbol{V} = -\frac{1}{\rho}\nabla p \qquad (3.1)$$

$$\frac{D\boldsymbol{V}}{Dt} + f\,\boldsymbol{k}\,x\boldsymbol{V} = -\nabla_p \phi \qquad (3.2)$$

where $\boldsymbol{V} = u\boldsymbol{i} + v\boldsymbol{j}$ is the horizontal velocity vector and ∇_p is the horizontal gradient operator applied with pressure held constant.

Because *p* is the independent vertical coordinate, we must expand the total derivative as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{Dx}{Dt}\frac{\partial}{\partial x} + \frac{Dy}{Dt}\frac{\partial}{\partial y} + \frac{Dp}{Dt}\frac{\partial}{\partial p} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p} \qquad (3.3)$$

Here $\omega = Dp/Dt$ is the "omega vertical motion," which is defined as the pressure change following the motion, similar to $w = Dz/Dt$. For synoptic motions, $\omega \approx -\rho g w$.

- From (3.2), the geostrophic relationship can be written as

$$f\,\boldsymbol{V_g} = \boldsymbol{k}\,x\nabla_p \phi \qquad (3.4)$$

or in scalar form

$$fu_g = -\frac{\partial \phi}{\partial y}, \quad fv_g = \frac{\partial \phi}{\partial x}. \qquad (3.4a\text{-}b)$$

Note there is no density present in (3.4). In addition, on an *f*-plane (i.e., *f* is constant),
we have

$$\nabla_p \cdot \boldsymbol{V_g} = 0$$

That is, there is no divergence for the geostrophic flow (non-divergent).

- The continuity equation in the isobaric coordinates becomes

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p + \frac{\partial \omega}{\partial p} = 0 \quad . \tag{3.5}$$

- The thermodynamic energy equation in the isobaric coordinates becomes

$$c_p \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} = J \tag{2.42}$$

then becomes

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - S_p\omega = \frac{J}{c_p} \tag{3.6}$$

where $J = \frac{Dq}{Dt}$ is the diabatic heating rate and

$$S_p \equiv \frac{RT}{c_p p} - \frac{\partial T}{\partial p} = -\frac{T}{\theta}\frac{\partial \theta}{\partial p}, \text{ or } \quad S_p \equiv \frac{\Gamma_d - \Gamma}{\rho g} \tag{3.7}$$

where S_p is the "static stability parameter".