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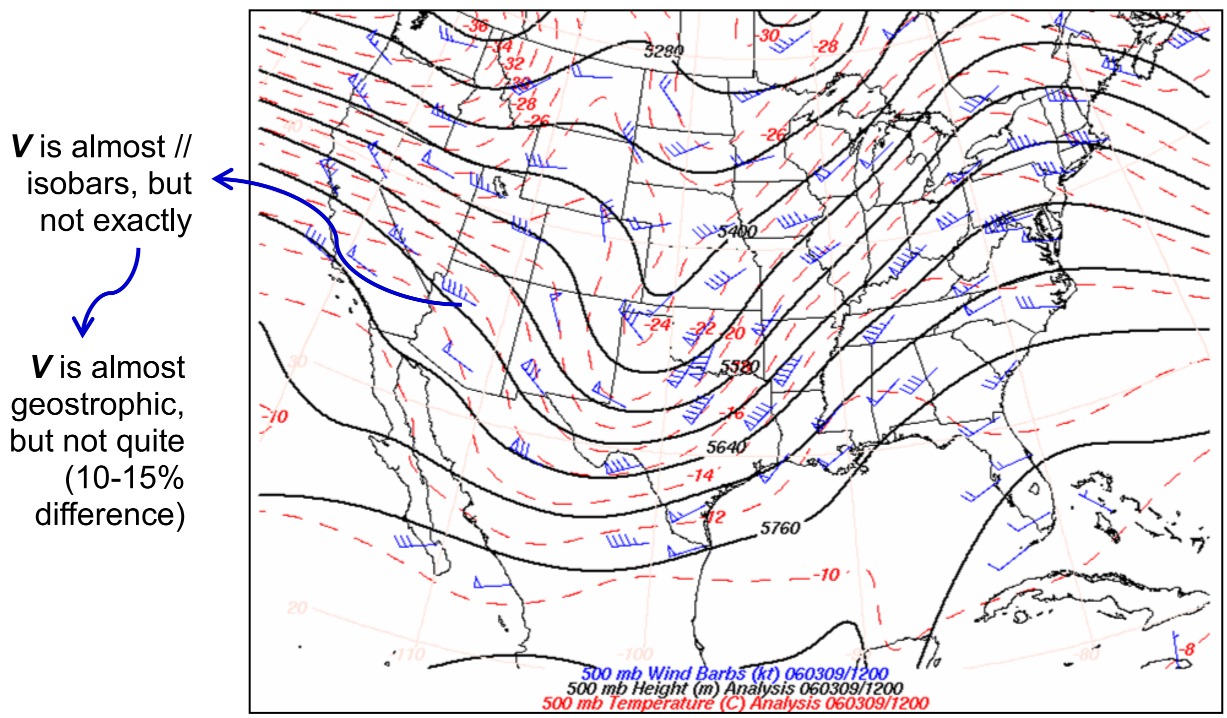
Chapter 7 Quasi-Geostrophic (QG) Approximation and QG Vorticity Equation

7.1 Quasi-Geostrophic Approximation

Purpose: The primitive equations contain several terms of secondary significance, thus can be simplified by making the quasi-geostrophic (QG) approximation to help understand the basic dynamics of synoptic atmospheric motions in midlatitude.



QG approximation will lead to two major equations:

- Geopotential height tendency equation for predicting the height tendency
- Omega equation for diagnose the vertical motion

➢ QG Approximation of the momentum equations

Consider the primitive equations in isobaric coordinates

$$\frac{Du}{Dt} = fv - \frac{\partial \phi}{\partial x}$$ *x*-momentum equation

$$\frac{Dv}{Dt} = -fu - \frac{\partial \phi}{\partial y}$$ *y*-momentum equation

$$\frac{\partial \phi}{\partial p} = -\frac{RT}{p}$$ Hydrostatic equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$ Continuity equation

$$\frac{DT}{Dt} = \left(\frac{RT}{c_p p}\right)\omega + \left(\frac{1}{c_p}\right)\frac{DQ}{Dt}$$ Thermodynamic equation

➢ We have already made the following assumptions or approximations:
(1) Inviscid
(2) Hydrostatic
(3) No Earth curvature effect

➢ Next, the total horizontal velocity is decomposed into geostrophic and ageostrophic components

$u = u_g + u_a$ and $v = v_g + v_a$, or in vector form

$\boldsymbol{V} = \boldsymbol{V}_g + \boldsymbol{V}_a$

(4) Assume $u_a << u_g$ and $v_a << v_g$, which gives

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \omega\frac{\partial u}{\partial p} \approx \frac{\partial u_g}{\partial t} + u_g\frac{\partial u_g}{\partial x} + v_g\frac{\partial u_g}{\partial y} = \frac{D_g u_g}{Dt},$$

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \omega\frac{\partial v}{\partial p} \approx \frac{\partial v_g}{\partial t} + u_g\frac{\partial v_g}{\partial x} + v_g\frac{\partial v_g}{\partial y} = \frac{D_g v_g}{Dt},$$

where $\frac{D_g}{Dt} = \frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y}$.

It is valid only when R_o<<1, where is the $R_o=U/fL$ is the Rossby number.

(5) Make midlatitude β-plane approximation:

$$f = f_o + \frac{\partial f}{\partial y}y + \frac{1}{2!}\frac{\partial^2 f}{\partial y^2}y^2 + \cdots \approx f_o + \frac{\partial f}{\partial y}y = f_o + \beta y$$

where β (=$\partial f/\partial y$) is a constant and y is the meridional distance from the latitude for f_o.

(6) Further assume $f = f_o$ in the geostrophic wind balance equations,

$$u_g \equiv -\frac{1}{f_o}\frac{\partial \phi}{\partial y}, \quad v_g \equiv \frac{1}{f_o}\frac{\partial \phi}{\partial x}$$

With the above assumptions and approximations, the x-momentum equation becomes

$$\frac{\partial u_g}{\partial t} + u_g \frac{\partial u_g}{\partial x} + v_g \frac{\partial u_g}{\partial y} = -\frac{\partial \phi}{\partial x} + (f_0 + \beta y)(v_g + v_a) \approx f_0 v_a + \beta y v_g$$

$$\frac{\partial v_g}{\partial t} + u_g \frac{\partial v_g}{\partial x} + v_g \frac{\partial v_g}{\partial y} = -\frac{\partial \phi}{\partial y} - (f_0 + \beta y)(u_g + u_a) \approx -f_0 u_a - \beta y u_g$$

In summary, the QG approximation neglects the following effects:

- Friction
- Horizontal advection of momentum by the ageostrophic wind, e.g. $\left(u_a \frac{\partial u_g}{\partial x}, v_a \frac{\partial u_g}{\partial y}\right)$, & on ageostrophic wind, e.g. $\left(u_a \frac{\partial u_a}{\partial x}, v_a \frac{\partial u_a}{\partial y}\right)$
- Vertical advection of momentum, e.g. $\left(\omega \frac{\partial u_g}{\partial p}, \omega \frac{\partial u_a}{\partial p}\right)$
- Local changes in the ageostrophic wind, e.g. $\frac{\partial u_a}{\partial t}$
- Advection of the ageostrophic momentum by the geostrophic wind, e.g. $\left(u_g \frac{\partial u_a}{\partial x}, v_g \frac{\partial u_a}{\partial y}\right)$

➢ QG Continuity Equation

Substituting $u = u_g + u_a$ and $v = v_g + v_a$ into

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

as earlier, assuming u_a, v_a, and ω are one order smaller than that of u_g and v_g, and using

$u_g \equiv -\frac{1}{f_o}\frac{\partial \phi}{\partial y}$, $v_g \equiv \frac{1}{f_o}\frac{\partial \phi}{\partial x}$, and $\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0$, lead to

$$\frac{\partial u_a}{\partial x}+\frac{\partial v_a}{\partial y}+\frac{\partial \omega}{\partial p}=0$$

➢ QG Thermodynamic Equation

The thermodynamic energy equation in the primitive set of equations [see ASME 433 note Eq. (3.6)]

$$\frac{DT}{Dt}=\left(\frac{RT}{c_p p}\right)\omega+\left(\frac{1}{c_p}\right)\frac{DQ}{Dt}$$

can be rewritten as

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}-\left(\frac{\sigma p}{R}\right)\omega=\frac{1}{c_p}\frac{DQ}{Dt}$$ where $\sigma=-\frac{RT}{\theta p}\frac{\partial \theta}{\partial p}$.

Applying the primary QG approximation ($u \approx u_g$ and $v \approx v_g$) leads to

$$\frac{\partial T}{\partial t}=-u_g\frac{\partial T}{\partial x}-v_g\frac{\partial T}{\partial y}+\left(\frac{\sigma p}{R}\right)\omega+\frac{J}{c_p}$$

In summary, the QG equations can be written as

$$\frac{\partial u_g}{\partial t}+u_g\frac{\partial u_g}{\partial x}+v_g\frac{\partial u_g}{\partial y}=f_0 v_a+\beta y v_g \qquad (6.10)$$

$$\frac{\partial v_g}{\partial t}+u_g\frac{\partial v_g}{\partial x}+v_g\frac{\partial v_g}{\partial y}=-f_0 u_a-\beta y u_g \qquad (6.11)$$

$$\frac{\partial \phi}{\partial p}=-\frac{RT}{p} \qquad (6.2)$$

$$\frac{\partial u_a}{\partial x}+\frac{\partial v_a}{\partial y}+\frac{\partial \omega}{\partial p}=0 \qquad (6.12)$$

$$\frac{\partial T}{\partial t}=-u_g\frac{\partial T}{\partial x}-v_g\frac{\partial T}{\partial y}+\left(\frac{\sigma p}{R}\right)\omega+\frac{J}{c_p} \qquad (6.13)$$

$$u_g = -\frac{1}{f_0}\frac{\partial \phi}{\partial y}$$
$$v_g = \frac{1}{f_0}\frac{\partial \phi}{\partial x} \qquad (6.7)$$

7.2 Quasi-Geostrophic Vorticity Equation

➢ Start with QG equations of motion

$$\frac{\partial u_g}{\partial t} + u_g \frac{\partial u_g}{\partial x} + v_g \frac{\partial u_g}{\partial y} = f_0 v_a + \beta y v_g \qquad (6.10)$$

$$\frac{\partial v_g}{\partial t} + u_g \frac{\partial v_g}{\partial x} + v_g \frac{\partial v_g}{\partial y} = -f_0 u_a - \beta y u_g \qquad (6.11)$$

Taking cross differentiation of the above equations leads to the QG vorticity equation

$$\frac{\partial \zeta_g}{\partial t} + u_g \frac{\partial \zeta_g}{\partial x} + v_g \frac{\partial \zeta_g}{\partial y} = f_0 \frac{\partial \omega}{\partial p} - \beta v_g \qquad (6.18)$$

where

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_o}\nabla^2 \phi \qquad (6.15)$$

Equation (6.18) can be rewritten as

$$\frac{\partial \zeta_g}{\partial t} = -V_g.\nabla(\zeta_g + f) + f_0 \frac{\partial \omega}{\partial p} \qquad (6.18)'$$

Physical meaning of (6.18)’.

LHS: local rate of change of geostrophic relative vorticity

RHS: (1) advection of absolute vorticity by geostrophic wind
(2) vorticity stretching by planetary vorticity

Substituting (6.12) into the equation leads to an alternative form of the QG vorticity equation

$$\frac{\partial \zeta_g}{\partial t} = -u_g \frac{\partial \zeta_g}{\partial x} - v_g \frac{\partial \zeta_g}{\partial y} - f_0\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) - \beta v_g \quad (6.18)$$

Thus, ζ_g can be predicted if ϕ and the right-hand side of is known.