Mesoscale Dynamics

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Chapter 2 Governing equations for mesoscale motions

2.1 Introduction

In this chapter, we will derive the governing equations for a stratified inviscid atmosphere on an *f* plane. The equations are based on the following physical laws: (a) Newton's second law of motion, (b) conservation of mass, and (c) first law of thermodynamics. These laws are represented by the set of primitive equations that are comprised by the horizontal and vertical momentum equations, the continuity equation, and the thermodynamic energy equation. Several approximations used in simplifying the set of primitive equations will be discussed as well as appropriate upper and lower boundary conditions. It should be mentioned that wave motions behave completely differently from mass transport. Briefly speaking, fluid particles do not necessarily follow the disturbance in a wave motion, but they do always follow the mass transport process. For example, air parcels associated with gravity waves may oscillate in the vicinity of the source or forcing region, but the gravity waves themselves may propagate to great distances from their origin. On the other hand air parcels, within a cold pool generated by evaporative cooling associated with falling raindrops beneath a thunderstorm, always move in concert with the density current.

2.2 Derivation of the governing equations

Considering an atmosphere on a planetary f plane, the momentum equations, continuity equation, and thermodynamic energy equation can be expressed in the following form:

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_{rx}, \tag{2.2.1}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_{ry}, \tag{2.2.2}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_{rz}, \tag{2.2.3}$$

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0, \tag{2.2.4}$$

$$\frac{D\theta}{Dt} = \frac{\theta}{c_p T}\dot{q}, \tag{2.2.5}$$

where $D/Dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y + w\partial/\partial z$ is the total (material) derivative, which represents the rate of change of a certain property within a fluid parcel following the motion, and $F_{rx}, F_{ry},$ and F_{rz} are the viscous terms or frictional forces per unit mass in the *x*, *y*, and *z* directions, respectively. The symbol c_p denotes the heat capacity of dry air at constant pressure, and $\dot{q}$ is the diabatic heating rate per unit mass in J kg^{-1} s^{-1}. Other symbols are defined as usual (also see Appendix A). In the *viscous sublayer*, which is a very thin layer near the Earth's surface, the viscous terms may be represented by molecular viscosity in the form $\nu\nabla^2 u, \nu\nabla^2 v,$ and $\nu\nabla^2 w$, where ν is the kinematic viscosity coefficient associated with molecular viscosity. Note that ν is equal to μ/ρ, where μ is the dynamic viscosity coefficient and ρ is the air density. At sea level, ν has a value of about 1.46 x 10^{-6} m^{2}s^{-1}. The molecular viscosity is almost completely negligible in the atmosphere above the viscous sublayer, where momentum and heat transfers are dominated by turbulent eddy motion. A number of parameterization schemes for turbulent eddy viscosity in the planetary boundary layer have been proposed in the literature and can be found in many numerical weather prediction textbooks. Some of these schemes will be discussed in Chapter 14.

The equation set (2.2.1) - (2.2.3) with no Coriolis term is often referred to as the *Navier-Stokes equations of motion*. The diabatic heating rate may be taken to represent, for example, surface sensible heating, elevated latent heating, or cloud-top radiative cooling. Note that the viscous terms on the right-hand side of (2.2.1) and (2.2.2) can be approximated by *Rayleigh friction* ($-\nu_o u$, $-\nu_o v$), while the diabatic heating term of (2.2.5) can be approximated by the *Newtonian cooling* ($-\nu_o \theta$), as is practiced in some theoretical studies to simplify the above system of governing equations. The coefficient ν_o is determined by the e-folding time scale of the disturbance. In the following, we will assume that the fluid is inviscid. In the above system (2.2.1) - (2.2.5), there are seven unknowns represented by five equations. In order to close the system, we need two additional equations, such as the equation of state for dry air (which is well represented by an ideal gas),

$$p = \rho R_d T \,, \tag{2.2.6}$$

and the Poisson's equation

$$\theta = T\left(\frac{p_s}{p}\right)^{R_d/c_p}, \tag{2.2.7}$$

where θ is the potential temperature, p_s is a constant reference pressure level (normally chosen as 1000 hPa) and R_d is the gas constant for dry air. For a moist atmosphere, the temperature in (2.2.6) is replaced by the virtual temperature, which takes into account the moist effects due to latent heat release, and the density is replaced by the total density, which is a sum of the dry air density and the total water density.

It should be noted that the continuity equation, (2.2.4), contains the implicit assumption that atmospheric mass is conserved. However, this assumption is violated when precipitation forms and falls to the surface. Similarly, evaporation from the surface adds atmospheric mass. Scale analysis indicates that the mass source/sink term, which should appear on the right side of (2.2.4) (e.g. Dutton 1986 - Section 8.1) is usually much smaller than other terms in the equation, but can be important in tropical cyclones or other heavily precipitating systems. The precipitation/evaporation mass sink/source term can play a non-negligible role in the dynamics of such systems, and should be included in numerical models used to simulate them (e.g., Lackmann and Yablonsky 2004).

The above equation set may be linearized by partitioning the field variables:

$$u(t,x,y,z) = U(z) + u'(t,x,y,z),$$

$$v(t,x,y,z) = V(z) + v'(t,x,y,z),$$

$$w(t,x,y,z) = w'(t,x,y,z),$$

$$\rho(t,x,y,z) = \bar{\rho}(x,y,z) + \rho'(t,x,y,z),$$

$$p(t,x,y,z) = \bar{p}(x,y,z) + p'(t,x,y,z),$$

$$\theta(t,x,y,z) = \bar{\theta}(x,y,z) + \theta'(t,x,y,z),$$

$$T(t,x,y,z) = \bar{T}(x,y,z) + T'(t,x,y,z),$$

$$\dot{q}(t,x,y,z) = q'(t,x,y,z)\,, \tag{2.2.8}$$

where capital letters and overbars represent the basic state, such as synoptic scale flow in which the mesoscale disturbances evolve, and the primes indicate perturbations from the basic state, such as the mesoscale flow fields. The basic state is assumed to follow Newton's second law of motion, conservation of mass, and the first law of thermodynamics. The basic state is assumed to be in equilibrium. That is, it is in geostrophic balance,

$$U = -\frac{1}{f\bar{\rho}}\frac{\partial \bar{p}}{\partial y} \quad \text{and} \quad V = \frac{1}{f\bar{\rho}}\frac{\partial \bar{p}}{\partial x}\,, \tag{2.2.9}$$

and in hydrostatic balance,

$$\frac{\partial \bar{p}}{\partial z} = -\bar{\rho} g\,, \tag{2.2.10}$$

where $\bar{p} = \bar{\rho} R_d \bar{T}$. Equations (2.2.9) and (2.2.10) automatically imply *thermal wind balance* for the basic state

$$U_z = -\frac{g}{f\bar{\theta}}\frac{\partial \bar{\theta}}{\partial y}\,;\; V_z = \frac{g}{f\bar{\theta}}\frac{\partial \bar{\theta}}{\partial x}\,, \tag{2.2.11}$$

where $\bar{\theta} = \bar{T}\left(p_o / \bar{p}\right)^{R_d / c_p}$ and subscripts indicate partial differentiations.

Conservation of mass, (2.2.4), of the basic state leads to

$$\frac{D\bar{\rho}}{Dt} + \bar{\rho}\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) = \frac{1}{c_s^2}\frac{D\bar{p}}{Dt} = \frac{1}{c_s^2}\left(U\frac{\partial \bar{p}}{\partial x} + V\frac{\partial \bar{p}}{\partial y}\right) = 0\,, \tag{2.2.12}$$

where c_s is the sound wave speed defined as $\sqrt{\gamma R_d \bar{T}}$. The last equality of (2.2.12) is consistent with the geostrophic wind relation. Conservation of the basic state thermal energy gives

$$U\frac{\partial\bar{\theta}}{\partial x}+V\frac{\partial\bar{\theta}}{\partial y}=0\,, \qquad (2.2.13)$$

which implies no basic state thermal advection by the basic wind, and will be assumed for deriving the perturbation thermodynamic equation. The left hand side of (2.2.13) is required to satisfy the constraint that the vertical motion field vanishes at the surface and possibly at the upper boundary for some theoretical studies (Bannon 1986). In the *Eady* (1949) *model* of baroclinic instability, this term is assumed to be 0. In fact, if one assumes $V=0$, then the above equation is automatically satisfied because $\partial\bar{\theta}/\partial x=(f\bar{\theta}/g)V_z=0$, based on the basic-state thermal wind relations. Substituting (2.2.8) with (2.2.9) - (2.2.13) into (2.2.1) - (2.2.5) and neglecting the nonlinear and viscous terms, the perturbation equations for mesoscale motions in the free atmosphere (i.e. above the planetary boundary layer) can be obtained,

$$\frac{\partial u'}{\partial t}+U\frac{\partial u'}{\partial x}+V\frac{\partial u'}{\partial y}+U_z w'-fv'+\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}=0\,, \qquad (2.2.14)$$

$$\frac{\partial v'}{\partial t}+U\frac{\partial v'}{\partial x}+V\frac{\partial v'}{\partial y}+V_z w'+fu'+\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial y}=0\,, \qquad (2.2.15)$$

$$\frac{\partial w'}{\partial t}+U\frac{\partial w'}{\partial x}+V\frac{\partial w'}{\partial y}-g\frac{\theta'}{\bar{\theta}}+\frac{p'}{\bar{\rho}H}+\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z}=0\,, \qquad (2.2.16)$$

$$\frac{1}{c_s^2}\left(\frac{\partial p'}{\partial t}+U\frac{\partial p'}{\partial x}+V\frac{\partial p'}{\partial y}+\bar{\rho}f(Vu'-Uv')\right)-\frac{\bar{\rho}}{H}w'+\bar{\rho}\nabla\cdot\boldsymbol{V'}=\frac{\bar{\rho}}{c_p\bar{T}}q'\,, \qquad (2.2.17)$$

$$\left(\frac{\partial\theta'}{\partial t}+U\frac{\partial\theta'}{\partial x}+V\frac{\partial\theta'}{\partial y}\right)+\frac{f\bar{\theta}}{g}(V_z u'-U_z v')+\frac{N^2\bar{\theta}}{g}w'=\frac{\bar{\theta}}{c_p\bar{T}}q'\,, \qquad (2.2.18)$$

where N is the buoyancy (Brunt-Vaisala) frequency and H is the scale height. The buoyancy frequency and scale height are defined, respectively, as

$$N^2 \equiv \frac{g}{\overline{\theta}}\frac{\partial\overline{\theta}}{\partial z},\ \ H \equiv \frac{c_s^2}{g}, \tag{2.2.19}$$

where $\gamma = c_p / c_v$. Note that the *scale height* has also been defined in the literature as the height at which the basic density of the air at surface (ρ_s) is reduced to its e-folding value, i.e. $\rho(z = H) = \rho_s e^{-1}$, assuming the air density decreases with height exponentially.

In deriving (2.2.14) and (2.2.15), we have assumed $|\rho'/\overline{\rho}| << |u'/U|$ and $|\rho'/\overline{\rho}| << |v'/V|$, which is a good first approximation in the real atmosphere. The sum of the fourth and fifth terms of (2.2.16) represents the buoyancy force associated with the atmospheric motion, which is equal to $g\rho'/\overline{\rho} = -g\theta'/\overline{\theta} + p'/(\overline{\rho}H)$. This relation reduces to $\rho'/\overline{\rho} \approx -\theta'/\overline{\theta}$ for an incompressible or Boussinesq fluid. The incompressible and Boussinesq approximations will be discussed in Section 2.3. The buoyancy frequency represents the natural oscillation frequency of an air parcel displaced vertically from its equilibrium position by the buoyancy force in a stably stratified atmosphere (i.e. $N^2 > 0$). The vertical oscillation period for parcels in this type of atmosphere is $2\pi / N$. In deriving (2.2.17), we have substituted the equation of state and the first law of thermodynamics, $D(\ln\theta)/Dt = \dot{q}/(c_p T)$, into the total derivative of the air density ($D\rho/Dt$). This yields the diabatic term on the right hand side of (2.2.17), which may be neglected for an incompressible or Boussinesq fluid since it is of the same order as other $\overline{c}_s^2$ terms for most mesoscale flows. The continuity equation reduces to

$$\frac{\partial\rho'}{\partial t} + \boldsymbol{V}\cdot\nabla\rho' + w\frac{d\overline{\rho}}{dz} + (\overline{\rho}+\rho')\nabla\cdot\boldsymbol{V} = 0, \tag{2.2.20}$$

where $\boldsymbol{V} = (u, v, w)$. For small-amplitude perturbations, the above equation reduces to the following linear form:

$$\frac{1}{\bar{\rho}}\frac{\partial \rho'}{\partial t} + \frac{1}{\bar{\rho}}\bar{\boldsymbol{V}} \cdot \nabla \rho' + \frac{w}{\bar{\rho}}\frac{d\bar{\rho}}{dz} + \nabla \cdot \boldsymbol{V}' = 0 . \qquad (2.2.21)$$

To formulate a more complete atmospheric system, we need to include nonlinear advective accelerations, viscosity and conservation equations for water substances (e.g. water vapor, cloud water, rain, ice, snow, and hail).

2.3 Approximations to the governing equations

Equations (2.2.14)-(2.2.18) form an elastic fluid system that may include the following types of waves:

(1) pure acoustic waves: c_s finite, $g = 0$, $f = 0$,

(2) acoustic-gravity waves: c_s/g finite, $f = 0$,

(3) pure gravity waves: $c_s \to \infty$, $g \neq 0$, $f = 0$, and

(4) inertia-gravity waves: $c_s \to \infty$, $g \neq 0$, $f \neq 0$.

In general, the system of (2.2.1)-(2.2.7) may include static (buoyant), shear (Kelvin-Helmholtz), symmetric, inertial, and baroclinic instabilities. Wave and instability dynamics will be discussed in later chapters. Our purpose here is to discuss some commonly used approximations and their limitations with regard to understanding the dynamics of real, observable mesoscale systems and circulations.

One of the simplest approximations is to assume that the scale height (H) and the adiabatic sound wave speed (c_s) are both independent of height. This corresponds to making the assumption of an isothermal atmosphere. It is a good first approximation, since the temperature in the troposphere, the layer in which most of the weather phenomena occur, varies vertically by about 20% (although somewhat more in the stratosphere). Another approximation that has often been adopted by meteorologists is the anelastic approximation. This approximation can be made by simply setting the terms involving $\bar{c}_s^2$ equal to zero in the continuity equation, (2.2.17), while keeping H finite. The effect of this approximation is to eliminate all waves with very high propagation speeds associated with rapid (adiabatic) compression and expansion of the fluid. In this approximation, the continuity equation then becomes

$$\nabla \cdot \boldsymbol{V}' - \frac{w'}{H} = 0, \tag{2.3.1}$$

or

$$\nabla \cdot (\boldsymbol{V}' e^{-z/H}) = 0, \tag{2.3.2}$$

since the scale height is taken to be a constant. Equations (2.3.1) and (2.3.2) may also be expressed in an alternate form:

$$\nabla \cdot (\bar{\rho} \boldsymbol{V}') = 0. \tag{2.3.3}$$

Note that (2.3.3) is linked with (2.3.2) when the density decays exponentially with height. Equation (2.3.1), (2.3.2), or (2.3.3) is called the *anelastic* or *deep convection continuity equation*. Equation (2.3.3) was first proposed by Batchelor (1953), who defined $\bar{\rho}(z)$ to be the density in an adiabatic, stably stratified, horizontally uniform reference state. The name anelastic was coined by Ogura and Phillips (1962), who derived (2.3.3) through a rigorous scale analysis, along with approximate forms for the momentum and thermodynamic energy equations. Their scaling analysis assumes that: (a) all deviations of the potential temperature θ' from some constant mean value θ_o are small and (b) the time scale of the disturbance is comparable to the time scale for gravity wave oscillations. The terms that are neglected in the original anelastic equations are an order $\varepsilon\ (=\theta\ '/\theta_o)$ smaller than those that are retained. Thus, in the case of dry convection (where mixing will keep the environmental lapse rate close to the adiabatic lapse rate), ε will be small and the anelastic equations can be used to represent nonacoustic modes with complete confidence. For deep, moist convection or gravity wave propagation, however, the mean-state static stability can be sufficient to make ε rather large. For example, the θ' variations across a 10 km deep isothermal layer may reach as high as 40% of the mean θ_o

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Equation (2.3.1) may be further simplified, by assuming that the vertical scale (L_z) of the disturbance is much smaller than the scale height of the basic state atmosphere, $L_z / H << 1$,

$$\nabla \cdot \boldsymbol{V'} = 0 . \qquad (2.3.4)$$

The above equation is called the *incompressible* or *shallow convection continuity equation*. This means that conservation of mass has become conservation of volume because density is treated as a constant. Thus, volume is a good proxy for mass under this approximation. It is important to distinguish the difference between the scale height (H) and the vertical scale of the disturbance or convection (L_z) because the scale height is controlled by the basic structure of the atmosphere, instead of by the fluid motion. Anelastic and incompressible approximations to the continuity equation can also be obtained by applying scale analysis to (2.2.17).

To improve the accuracy of the anelastic or incompressible hydrodynamic equations, the *pseudo-incompressible approximation* has been proposed (Durran 1989; Nance and

Durran 1994). Under the pseudo-incompressible approximation, the set of equations governing inviscid, rotating flow may be written:

$$\frac{Du}{Dt} - fv + (c_p\theta)\frac{\partial \pi'}{\partial x} = 0, \tag{2.3.5}$$

$$\frac{Dv}{Dt} + fu + (c_p\theta)\frac{\partial \pi'}{\partial y} = 0, \tag{2.3.6}$$

$$\frac{Dw}{Dt} + (c_p\theta)\frac{\partial \pi'}{\partial z} = g\frac{\theta'}{\overline{\theta}}, \tag{2.3.7}$$

$$\nabla \cdot (\overline{\rho}\overline{\theta}\, \mathbf{V}) = \frac{\dot{Q}}{c_p\overline{\pi}}, \tag{2.3.8}$$

$$\frac{D\theta}{Dt} = \frac{\theta\dot{Q}}{c_p\overline{\pi}\,\overline{\rho}\,\overline{\theta}}, \tag{2.3.9}$$

where $\dot{Q}$ is the heating rate per unit volume [= $(\overline{\rho}\overline{\theta}/\theta)\dot{q}$] and π, the *Exner function*, is defined as

$$\pi = \left(\frac{p}{p_o}\right)^{R_d/c_p}. \tag{2.3.10}$$

The Exner function is partitioned into the basic state and the perturbation,

$$\pi = \overline{\pi}(z) + \pi'(t, x, y, z). \tag{2.3.11}$$

Note that the Exner function has been defined in different forms in the literature. In order to apply the pseudo-incompressible approximation in the investigation of mesoscale atmospheric motions, the following two criteria must be met: (1) $T >> L/c_s$, i.e. the Lagrangian time scale associated with the disturbance must be much larger than the time scale associated with (adiabatic) sound wave propagation and (2) $\pi' << \overline{\pi}$.

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A well-known approximation that has been widely used in theoretical studies is the *Boussinesq approximation*. For the fully nonlinear equations governing thermal convection in a compressible fluid, the conditions under which the Boussinesq approximation is applicable are: (a) the vertical dimension of the fluid motion is much less than any scale height and (b) the motion-induced fluctuations in density and pressure do not exceed, in order of magnitude, the total static variations of these quantities (Spiegel and Veronis 1960). In (2.2.14)-(2.2.18), the Boussinesq approximation is equivalent to assuming that (1) $L_z << H$, (2) density is treated as a constant except where it is coupled to gravity in the buoyancy term in the vertical momentum equation, and (3) $\bar{\rho}$ and $\bar{\theta}$ are replaced by ρ_o and θ_o, respectively, in all equations. Under the Boussinesq approximation, one may define the *perturbation buoyancy* ($b' = g\theta'/\theta_o$) and *kinematic pressure* (p'/ρ_o) to simplify the equation set. The Boussinesq approximation can be extended to a larger vertical scale of motion if the potential density and modified pressure are used in a low Rossby number flow (Janowitz 1977).

For a disturbance in which the horizontal scale is much larger than its vertical scale ($L_x >> L_z$), the vertical acceleration generally becomes very small compared to the vertical pressure gradient force and buoyancy force and therefore may be neglected. This leads to the *hydrostatic approximation*. By eliminating the vertical accelerations, the perturbation pressure p' and density ρ' are in hydrostatic balance,

$$\frac{\partial p'}{\partial z} - \left(\frac{g\bar{\rho}}{\bar{\theta}}\right)\theta' = 0. \qquad (2.3.12)$$

The above perturbation hydrostatic equation has a corresponding Boussinesq form,

$$\frac{\partial p'}{\partial z} - \left(\frac{g\rho_o}{\theta_o}\right)\theta' = 0. \qquad (2.3.13)$$

Limitations of the hydrostatic approximation may be investigated by assuming a two-dimensional (in *x* and *z*), non-rotating, Boussinesq fluid with a uniform basic flow (i.e. U = constant; V = 0). Under these constraints, the linear set of perturbation equations (2.2.14) - (2.2.18) may be simplified and combined into a single equation for the vertical velocity *w'*,

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2}\right) + N^2 \frac{\partial^2 w'}{\partial x^2} = 0 . \qquad (2.3.14)$$

It can be shown that the leftmost $\partial^2 w'/\partial x^2$ term is associated with the vertical acceleration term of the vertical momentum equation, i.e. the first two terms of (2.2.16). It can be shown by comparing the scales of $\partial^2 w'/\partial x^2$ and $\partial^2 w'/\partial z^2$ that in order to neglect the vertical acceleration term, $L_z / L_x << 1$ is required.

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Problems

2.1 Based on Poisson's equation and the equation of state, show that $g\rho'/\bar{\rho} \approx -g\theta'/\bar{\theta} + p'/\bar{\rho}H$, where $H \equiv \gamma R\bar{T}/g$ for a small-amplitude disturbance. You may assume that the basic state satisfies Poisson's equation and the equation of state.

2.2 Derive the linear set of equations (2.2.14) - (2.2.18) by substituting (2.2.8) - (2.2.13) into (2.2.1) - (2.2.5).

2.3 Obtain the analytical solution of the simplified governing equation of motion, $\partial u/\partial t = -\mu u$, with the initial value u_o, and determine the value of the coefficient of

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the Rayleigh friction if it takes 12 hours for u to reduce to its e-folding value, i.e. $u = u_o e^{-1}$.

2.4 Perform scale analysis of (2.2.17) by (a) identifying scales of individual terms; (b) estimating magnitudes of individual terms by assuming the following fluid flow system: $U = 10\ \text{ms}^{-1}$, $W = 1\ \text{ms}^{-1}$, $L_x = 10$ km , $L_z = 10$ km , $H = 10$ km , $f = 10^{-4}\ \text{s}^{-1}$, $\bar{\rho} = 1\ \text{kg m}^{-3}$, $q' = 1000\ \text{W m kg}^{-1}/10\,\text{km}$, $u' = 1\ \text{ms}^{-1}$, and $p' = 1$ mb ; (c) finding the approximate form of the equation by keeping the highest order terms; and (d) identifying the approximation. What kind of weather systems does it describe?

2.5 (a) Let $\rho(t,x,y,z) = \bar{\rho}(z) + \rho'(t,x,y,z)$ in (2.2.4) and identify scales of individual terms. Make sure to differentiate the vertical scale of the basic state (*H*) and that of disturbance (L_z) and the scales of individual terms of the divergence term. (b) Consider the following scales of a weather system: $U = 10\ \text{ms}^{-1}$, $W = 1\ \text{ms}^{-1}$, $L_x = 10$ km, $L_z = 10$ km, $H = 10$ km, $\bar{\rho} = 1\ \text{kg m}^{-3}$, and $\rho' = 0.01\ \text{kg m}^{-3}$. Show that the approximate continuity equation is reduced to $(w'/\bar{\rho})(\partial\bar{\rho}/\partial z) + (\partial u'/\partial x + \partial v'/\partial y + \partial w'/\partial z) = 0$ by retaining only the highest two orders of magnitudes. (c) Show that the approximate equation of (b) is identical to the anelastic equation (2.3.1) if the basic density decreases with height with an e-folding value of *H*. (d) Based on the scales identified in (a), derive the criterion for the shallow convection continuity equation (2.3.4).

2.6 Prove that (2.3.1), (2.3.2), and (2.3.3) are identical.

2.7 Apply the Boussinesq approximation to (2.2.14)-(2.2.18).

2.8 Derive (2.3.14) from the linear equations (2.2.14)-(2.2.18) by assuming a two-dimensional (in x and z), non-rotating, Boussinesq fluid with a uniform basic flow (i.e. U = constant; V = 0). Show that: (a) the $\partial^2 w' / \partial x^2$ term is associated with the vertical acceleration term of the vertical momentum equation, i.e. the first two terms of (2.2.16), and (b) taking $L_z / L_x << 1$ is equivalent to making the hydrostatic approximation.

2.9 (a) Derive the *Reynolds number* (*Re*) by taking the ratio of the scale of the *inertial term* (Du / Dt) and the molecular viscosity term ($\nu \nabla^2 u$). Assume that the viscous term can be approximated by the horizontal part of the *Laplacian operator*. Estimate this number at the *viscous sublayer* very near the earth's surface with $U = 10$ ms^{-1}, L = 120 km, and $\nu = 1.2x10^{-5}$m^{2}s^{-1}. What can you conclude?

(b) Assuming that the viscous term can be approximated by only the vertical derivative of the Laplacian operator, then use scale analysis to derive a new Reynolds number (Re_2). Assuming $Re_2 = Re_1$, estimate the new *eddy viscosity coefficient* (ν) by using Re_1, Re_2, U, L_z and L (where Re_1 is the Reynolds number from part a). What can you conclude?