Dynamical Mountain Meteorology

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(Ref.: *Mesoscale Dynamics*, Y.-L. Lin, Cambridge, 2007)

Chapter 7 Two-Dimensional Flow over Sinusoidal Mountains

(Based on Sec. 5.1, "Mesoscale Dynamics" by Y.-L. Lin 2007, Cambridge U. Press)
(Classical equation editor: $c_p \neq f(k)$)

- Many well-known weather phenomena are directly related to flow over orography, such as

  - mountain waves
  - lee waves and clouds
  - rotors and rotor clouds
  - severe downslope windstorms
  - lee vortices
  - lee cyclogenesis
  - frontal distortion across mountains
  - cold-air damming
  - track deflection of midlatitude and tropical cyclones
  - coastally trapped disturbances
  - orographically induced rain and flash flooding
  - orographically influenced storm tracks.

- A majority of these phenomena are mesocale and are induced by stably stratified flow over orography. Thus, understanding the dynamics associated with stably stratified flow over a mesoscale mountain is essential in improving the understanding of the above mentioned phenomena.

- In addition, understanding the dynamics of orographically forced flow will also help on different aspects of meteorology, such as turbulence which affect aviation safety, wind-damage risk assessment, pollution dispersion in complex terrain, and subgrid-scale parameterization of mountain wave drag in general circulation models.

5.1 Two-Dimensional Flow over Sinusoidal Mountains

[ref: 5.1 of Lin (2007)]

- Some fundamental properties of flow responses to orographic forcing can be understood by considering a two-dimensional, steady-state, adiabatic, inviscid, nonrotating, Boussinesq fluid flow over a small-amplitude mountain.

- For a two-dimensional, steady-state, adiabatic, inviscid, nonrotating, Boussinesq fluid flow, the linear governing equations (2.2.14) - (2.2.18)

$$\frac{\partial u'}{\partial t}+U\frac{\partial u'}{\partial x}+V\frac{\partial u'}{\partial y}+U_z w'-fv'+\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}=0, \tag{2.2.14}$$

$$\frac{\partial v'}{\partial t}+U\frac{\partial v'}{\partial x}+V\frac{\partial v'}{\partial y}+V_z w'+fu'+\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial y}=0, \tag{2.2.15}$$

$$\frac{\partial w'}{\partial t}+U\frac{\partial w'}{\partial x}+V\frac{\partial w'}{\partial y}-g\frac{\theta'}{\bar{\theta}}+\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z}+\frac{p'}{\bar{\rho}H}=0, \tag{2.2.16}$$

$$\frac{1}{c_s^2}\left(\frac{\partial p'}{\partial t}+U\frac{\partial p'}{\partial x}+V\frac{\partial p'}{\partial y}\right)-\frac{\overline{\rho}}{H}w'+\overline{\rho}\nabla\cdot\boldsymbol{V}'=\frac{\overline{\rho}}{c_p\overline{T}}q', \qquad (2.2.17)$$

$$\frac{\partial\theta'}{\partial t}+U\frac{\partial\theta'}{\partial x}+V\frac{\partial\theta'}{\partial y}+\frac{N^2\overline{\theta}}{g}w'=\frac{\overline{\theta}}{c_p\overline{T}}q', \qquad (2.2.18)$$

can be simplified to

$$U\frac{\partial u'}{\partial x}+U_z w'+\frac{1}{\rho_o}\frac{\partial p'}{\partial x}=0, \qquad (5.1.1)$$

$$U\frac{\partial w'}{\partial x}-g\frac{\theta'}{\theta_o}+\frac{1}{\rho_o}\frac{\partial p'}{\partial z}=0, \qquad (5.1.2)$$

$$\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z}=0, \qquad (5.1.3)$$

$$U\frac{\partial\theta'}{\partial x}+\frac{N^2\theta_o}{g}w'=0. \qquad (5.1.4)$$

- The above set of equations can be further reduced to *Scorer's equation* (1954),

$$\nabla^2 w'+l^2(z)w'=0, \qquad (5.1.5)$$

where $\nabla^2=\partial^2/\partial x^2+\partial^2/\partial z^2$ is the two-dimensional Laplacian operator, and l is the Boussinesq form of the *Scorer parameter* (Scorer 1949), which is defined as:

$$l^2(z)=\frac{N^2}{U^2}-\frac{U_{zz}}{U}. \qquad (5.1.6)$$

Equation (5.1.5) serves as a central tool for numerous theoretical studies of small-amplitude, two-dimensional mountain waves.

It may also be interpreted as a vorticity equation upon being multiplied by U (Smith 1979).

- The first term, $U(w'_{xx} + w'_{zz})$, is the rate of change of vorticity following a fluid particle.
- The second term, $N^2 w'/U$, is the rate of vorticity production by buoyancy forces.
- The last term, $-U_{zz}w'$, represents the rate of vorticity production by the redistribution of the background vorticity (U_z).

In the extreme case of very small Scorer parameter, i.e. very weak stratification and/or basic wind has zero or constant shear, (5.1.5) reduces to the irrotational or potential flow,

$$\nabla^2 w' = 0. \qquad (5.1.7)$$

As discussed in Chapter 3 [(3.5.22)], the buoyancy force is negligible in this extreme case. If the forcing is symmetric in the basic flow direction, such as a cylinder in an unbounded fluid or a bell-shaped mountain in a half-plane, then the flow is symmetric. For this particular case, there is no drag produced on the mountain since the fluid is inviscid.

- In order to simplify the mathematics of the steady state mountain wave problem, one may assume that $U(z)$ and $N(z)$ are independent of height, and a sinusoidal terrain

$$h(x) = h_m \sin kx, \tag{5.1.8}$$

where h_m is the mountain height and k is the wave number of the terrain.

For an inviscid fluid flow, the lower boundary condition requires the fluid particles to follow the terrain, so that the streamline slope equals the terrain slope locally,

$$\frac{w}{u} = \frac{w'}{U+u'} = \frac{dh}{dx} \quad \text{at} \quad z = h(x). \tag{5.1.9}$$

- For a small-amplitude mountain, this leads to the linear lower boundary condition

$$w' = U\frac{dh}{dx} \quad \text{at} \quad z = 0. \tag{5.1.10}$$

or

$$w'(x,0) = U\, h_m k \cos kx \quad \text{at} \quad z = 0, \tag{5.1.11}$$

for flow over a sinusoidal mountain as described by (5.1.8).

Due to the sinusoidal nature of the forcing, it is natural to look for solutions in terms of sinusoidal functions,

$$w'(x,z) = w_1(z)\cos kx + w_2(z)\sin kx. \tag{5.1.12}$$

Substituting the above solution into (5.1.5) with a constant Scorer parameter leads to

$$w_{izz} + (l^2 - k^2)w_i = 0, \quad i = 1, 2\,. \tag{5.1.13}$$

As discussed in Chapter 3 [(3.5.7)], two cases are possible: (a) $l^2 < k^2$ and (b) $l^2 > k^2$.

- Case 1: $N/U < k$ or $Na/U < 2\pi$, where a is the terrain wavelength.

  Physically, this means that the basic flow has relatively weaker stability and stronger wind, or that the mountain is narrower than a certain threshold.

For example, to satisfy the criterion for a flow with U = 10 ms^{-1} and N = 0.01 s^{-1}, the wavelength of the mountain should be smaller than 6.3 km.

In fact, this criterion can be rewritten as $(a/U)/(2\pi/N) < 1$. The numerator, a/U, represents the advection time of an air parcel passing over one wavelength of the terrain, while the denominator, $2\pi/N$, represents the period of buoyancy oscillation due to stratification.

This means that the time an air parcel takes to pass over the terrain is less than it takes for vertical oscillation due to buoyancy force. In other words, buoyancy force plays a smaller role than the horizontal advection.

In this situation, (5.1.13) can be rewritten as

$$w_{izz} - (k^2 - l^2)w_i = 0, \quad i = 1, 2 . \qquad (5.1.14)$$

The solutions of the above second-order differential equation with constant coefficient may be obtained

$$w_i = A_i e^{\lambda z} + B_i e^{-\lambda z}, \quad i = 1, 2 , \qquad (5.1.15)$$

where

$$\lambda = \sqrt{k^2 - l^2} \,. \qquad (5.1.16)$$

Similar to that described in Section 3.4, the upper boundedness condition requires $A_i = 0$ because the energy source is located at $z = 0$.

Applying the lower boundary condition, (5.1.11), and the upper boundary condition ($A_i = 0$) to (5.1.15) yields

$$B_1 = U\,h_m k; \quad B_2 = 0 \,. \qquad (5.1.17)$$

This gives the solution,

$$w'(x,z) = w_1(z)\cos kx = U h_m k e^{-\sqrt{k^2 - l^2}\,z} \cos kx \,, \qquad (5.1.18)$$

The vertical displacement (η) is defined as $w' = D\eta / Dt$ which reduces to

$$w' = \frac{D\eta}{Dt} = U \frac{\partial \eta}{\partial x} \qquad (5.1.19)$$

for a steady-state flow.

Equation (5.1.18) can then be expressed in terms of η,

$$\eta = \frac{1}{U} \int_0^x w'\,dx = h_m \sin kx \, e^{-\sqrt{k^2 - l^2}\,z} \,. \qquad (5.1.20)$$

The above solution is sketched in Fig. 5.1a.

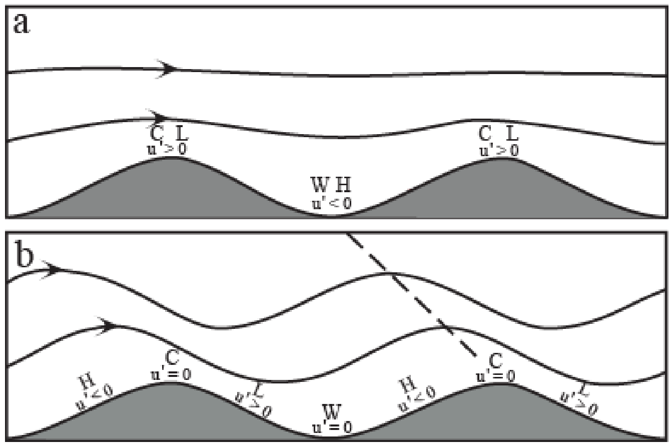

Fig. 5.1: The steady-state, inviscid flow over a two-dimensional sinusoidal mountain when (a) $l^2 < k^2$ (or $N < kU$), where k is the terrain wavenumber ($= 2\pi / a$, where a is the terrain wavelength), or (b) $l^2 > k^2$ (or $N > kU$). The dashed line in (b) denotes the constant phase line which tilts upstream with height. The maxima and minima of u', p' (H and L), and θ' (W and C) are also denoted in the figures.

The disturbance is symmetric with respect to the vertical axis and decays exponentially with height. Thus, the flow belongs to the evanescent flow regime as discussed in Section 3.5.

The buoyancy force plays a minor role compared to that of the advection effect. The other variables can also be obtained by using the governing equations and (5.1.18),

$$u' = U\, h_m \sqrt{k^2 - l^2}\, \sin kx\, e^{-\sqrt{k^2 - l^2}\, z}, \tag{5.1.21}$$

$$p' = -\rho_o U^2\, h_m \sqrt{k^2 - l^2}\, \sin kx\, e^{-\sqrt{k^2 - l^2}\, z}, \tag{5.1.22}$$

$$\theta' = -(\theta_o N^2 / g)\, h_m \sin kx\, e^{-\sqrt{k^2 - l^2}\, z}. \tag{5.1.23}$$

The maxima and minima of u', p', and θ' are also denoted in Fig. 5.1a.

- The coldest (warmest) air is produced at the mountain peak (valley) due to adiabatic cooling (warming).
- The flow accelerates over the mountain peaks and decelerates over the valleys.
- From the horizontal momentum equation, (5.1.1) with $U_z = 0$, or (5.1.22), a low (high) pressure is produced over the mountain peak (valley) where maximum (minimum) wind is produced.

Note that (5.1.1) is also equivalent to the *Bernoulli equation*, which states that the pressure perturbation is out of phase with the horizontal velocity perturbation.

Since no pressure difference exists between the upslope and downslope, this flow produces no net wave drag on the mountain (mountain drag).

The *mountain drag* can be computed either from the horizontal pressure force on the mountain over a wavelength,

$$D = \frac{k}{2\pi}\int_{-\pi/k}^{\pi/k} p'(x, z=0)\left(\frac{dh}{dx}\right)dx, \qquad (5.1.24)$$

or equivalently, as the negative of the vertical flux of horizontal momentum (*momentum flux*) in the wave motion,

$$\mathcal{D} = -\frac{\rho_o k}{2\pi}\int_{-\pi/k}^{\pi/k} u'w'\,dx\,. \qquad (5.1.25)$$

Note that the *Eliassen and Palm's theorem*, (4.4.10),

Quote from Ch.4 (Lin 2007):

$$F = \rho_o \int_{-\infty}^{\infty} u'w'\,dx = constant, \quad \text{when } U \neq 0\,. \qquad (4.4.9)$$

This is the *Eliassen and Palm* (1960) *theorem*, which states that the vertical flux of horizontal momentum does not change with height except possibly at levels where $U = 0$ or in the layer of forcing. If the integration of (4.4.8) is taken over one horizontal wavelength, we have

$$\overline{p'w'} = -\rho_o U \overline{u'w'}\,. \qquad (4.4.10)$$

Thus, the vertical flux of wave energy is negatively proportional to the vertical flux of horizontal momentum if $U > 0$. For an energy source located in the lower troposphere, such as a mountain, the momentum flux is downward because the energy flux is upward.

indicates that the vertical flux of horizontal momentum in a steady-state flow is negatively proportional to the vertical energy flux, $\overline{p'w'}$ (where the overbar denotes the average over a wavelength).

➢ Case 2: $l^2 > k^2$ ($N/U > k$ or $Na/U > 2\pi$)

As discussed in Section 3.5, this means that the basic flow has relatively stronger stability and weaker wind or that the mountain is wider.

For example, and as mentioned earlier, to satisfy the criterion for a flow with U = 10 ms^{-1}and N = 0.01 s^{-1}, the terrain wavelength should be larger than 6.3 km. Since $(a/U)/(2\pi/N) > 1$, the advection time is larger than the period of the vertical oscillation.

In other words, buoyancy force plays a more dominant role than the horizontal advection.

In this case, (5.1.13) can be written as

$$w_{izz} + m^2 w_i = 0, \quad m^2 = l^2 - k^2, \ i = 1, 2. \tag{5.1.26}$$

We look for solutions in the form

$$w_i(z) = A_i \sin mz + B_i \cos mz, \ i = 1, 2. \tag{5.1.27}$$

Substituting (5.1.27) into (5.1.12) leads to

$$\begin{aligned} w'(x,z) = C\cos(kx+mz) + D\sin(kx+mz) + \\ E\cos(kx-mz) + F\sin(kx-mz). \end{aligned} \tag{5.1.28}$$

In the above equation, terms of $(kx+mz)$ have an upstream phase tilt with height, while terms of $(kx-mz)$ have a downstream phase tilt.

It can be shown that terms of $(kx+mz)$ have a positive *vertical energy flux* and should be retained since the energy source in this case is located at the mountain surface. This satisfies the Sommerfeld radiation boundary condition (Sommerfeld 1949), as discussed in Section 4.4 (HW5). Thus, the solution requires $E=F=0$.

This flow regime is characterized as the upward propagating wave regime, as discussed in Chapter 3. As in the first case, the lower boundary condition requires

$$C=Uh_m k, \quad D=0. \tag{5.1.29}$$

This leads to

$$w'(x,z)=Uh_m k\cos(kx+mz). \tag{5.1.30}$$

Other variables can be obtained through definitions or the governing equations,

$$\eta(x,z)=h_m\sin(kx+mz), \tag{5.1.31}$$

$$u'(x,z)=-Uh_m m\cos(kx+mz), \tag{5.1.32}$$

$$p'(x,z) = \rho_o U^2 h_m m \cos(kx + mz) \text{, and} \tag{5.1.33}$$

$$\theta'(x,z) = -\frac{N^2 \theta_o h_m}{g} \sin(kx + mz) . \tag{5.1.34}$$

The vertical displacement of the flow, and the maxima and minina of u', p', and θ' are depicted in Fig. 5.1b.

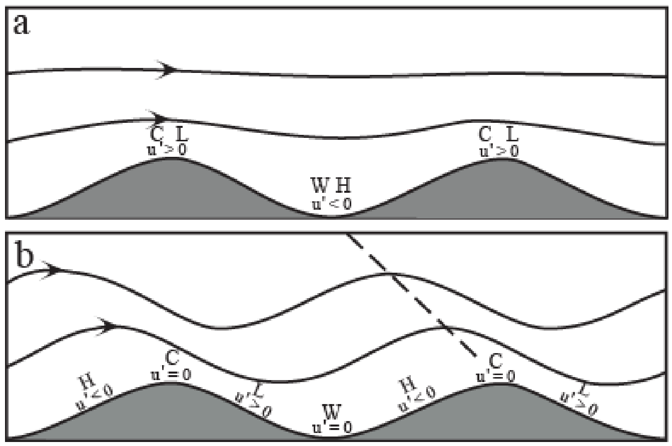

Fig. 5.1: The steady-state, inviscid flow over a two-dimensional sinusoidal mountain when (a) $l^2 < k^2$ (or $N < kU$), where k is the terrain wavenumber ($= 2\pi / a$, where a is the terrain wavelength), or (b) $l^2 > k^2$ (or $N > kU$). The dashed line in (b) denotes the constant phase line which tilts upstream with height. The maxima and minima of u', p' (H and L), and θ' (W and C) are also denoted in the figures.

Note that the flow pattern is no longer symmetric. The constant phase lines are tilted upstream (to the left) with height, thus producing a high pressure on the windward slope and a low pressure on the lee slope. Based on (5.1.32) or the Bernoulli equation (5.1.1), the flow decelerates over the windward slope and accelerates over the lee slope. The coldest and warmest spots are still located over the mountain peaks and valleys, respectively. The mountain drag can be calculated either from (5.1.24) or (5.1.25)

$$D = \frac{k}{2\pi}\int_{-\pi/k}^{\pi/k} p'(x, z=0)\left(\frac{dh}{dx}\right)dx, \tag{5.1.24}$$

$$\mathcal{D} = -\frac{\rho_o k}{2\pi}\int_{-\pi/k}^{\pi/k} u'w'dx. \tag{5.1.25}$$

to be

$$D = \frac{1}{2}\rho_o U^2 h_m^2 k\sqrt{l^2 - k^2}. \tag{5.1.35}$$

The positive wave drag on the mountain is produced by the high pressure on the windward slope and the low pressure on the lee slope. This also can be understood through (5.1.25) and the out-of-phase relationship of u' and w' over the windward and lee slopes, as shown in Fig. 5.1b.

- Extreme Case: $l^2 >> k^2$

  This means that buoyancy effect dominates and the advection effect is totally negligible.

In other words, the vertical pressure gradient force and the buoyancy force are roughly in balance and the vertical acceleration can be ignored.

Thus, the mountain waves become hydrostatic. In this limiting case, the governing equation becomes

$$w'_{zz} + l^2 w' = 0. \tag{5.1.36}$$

The flow pattern repeats itself in the vertical with a wavelength of $\lambda_z = 2\pi / l = 2\pi U / N$, which is also referred to as the hydrostatic vertical wavelength.

The regime boundary between the regimes of vertically propagating waves and evanescent waves can be found by letting $l = k$, which leads to $a = 2\pi U / N$.

The relation among the mountain waves discussed in this subsection is sketched in Fig. 5.2.

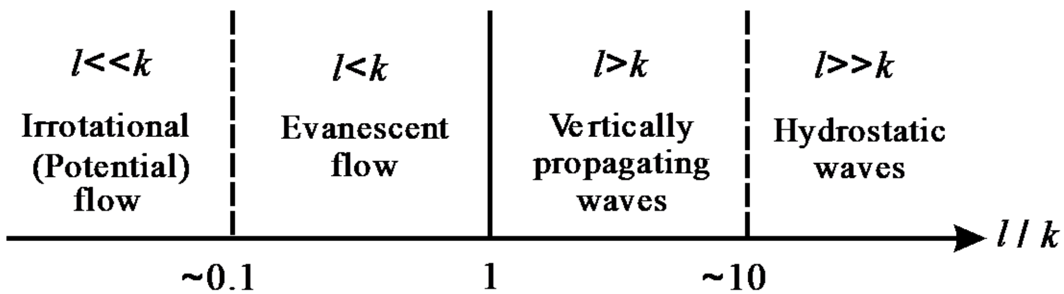

Fig. 5.2: Relations among different mountain wave regimes as determined by l/k, where l is the Scorer parameter and k is the wave number.